NONCOMMUTATIVE PLANE CURVES

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$k = \mathbb{C}$

PLANE CURVE

$f \in \mathbb{C}[x,y,z]_d$ homogeneous of degree d

$\rightsquigarrow \mathbb{V}(f) \subseteq \mathbb{P}^2 = \operatorname{Proj} \mathbb{C}[x,y,z]$

GOAL.
1) Define noncommutative plane $\mathbb{P}^2_{nc}$
2) Study analogue of $\mathbb{V}(f)$

I. NONCOMMUTATIVE PLANES

X, Y = smooth projective varieties

- [Gabriel 1962] $X \cong Y \iff \operatorname{coh}(X) \cong \operatorname{coh}(Y)$
- [Orlov 2001] If ω_X (anti)ample: $X \cong Y \iff D^b(X) \cong D^b(Y)$

Noncommutative algebraic geometry:
1) study $\mathcal{A}$ = abelian category which behaves like $\operatorname{coh}(X)$
2) study $\mathcal{T}$ = triangulated category which behaves like (admissible subcategory of) $D^b(X) = D^b(\operatorname{coh} X)$

Abelian categories which behave like coh X

1) $\operatorname{coh}(X, \mathcal{A})$ = (right) $\mathcal{A}$ - modules, coherent $/X$

$\mathcal{A}$ = coherent $\mathcal{O}_X$-algebra
(with $Z(\mathcal{A}) \supseteq \mathcal{O}_X$)

2) $\operatorname{coh}(\mathcal{X})$

stack with coarse moduli space X

3) $\operatorname{qgr}(A)$

graded $\mathbb{C}$-algebra

Le Bruyn (1995)
Chan-Ingalls (2004)
Faber-Ingalls-Okawa-Satriano (2022)

For curves & surfaces
there is a dictionary:
$\operatorname{coh}(X, \mathcal{A}) \simeq \operatorname{coh}(\mathcal{X}) \simeq \operatorname{qgr}(A)$

Idea (Serre 1955, Artin - Zhang 1994)

$\mathbb{C}[x,y,z]$ = homogeneous coordinate ring of $\mathbb{P}^2$

$$\operatorname{coh}(\mathbb{P}^2) \simeq \operatorname{qgr} \underbrace{\mathbb{C}[x,y,z]}_{\text{use noncommutative graded } \mathbb{C}\text{- algebra}} := \overset{\text{finitely generated graded } A\text{-modules}}{\operatorname{gr} \mathbb{C}[x,y,z]} \Big/ \underset{\text{finite-dimensional } A\text{-modules}}{\operatorname{fd} \mathbb{C}[x,y,z]}$$

DEFINITION (Artin - Schelter 1987)

$\underbrace{A_1 \cdot \ldots \cdot A_1}_{n \text{ times}} \subset A_n$ form a generating set of A_n (as a $\mathbb{C}$-vector space)

$\boxed{\dim A_1 < \infty}$

$A =$ $\overset{A_0 = \mathbb{C}}{\text{connected}}$ $\overset{A = \bigoplus_{n \in \mathbb{N}_0} A_n}{\text{graded}}$ $\mathbb{C}$-algebra generated in degree 1

is called 3-DIM QUADRATIC AS-REGULAR ALGEBRA if

i) gldim $A = 3$

ii) $\mathrm{Ext}^i_A(\underset{\mathbb{C} = A/A_{>0}}{\mathbb{C}}, A) \cong \begin{cases} \mathbb{C} & i = 3 \\ 0 & \text{else} \end{cases}$

iii) $h_A(t) = \dfrac{1}{(1-t)^3}$

In fact

$A \cong \mathbb{C}\langle x, y, z\rangle / (f_1, f_2, f_3)$

with $f_i \in \mathbb{C}\langle x, y, z\rangle$

EXAMPLE

0) $\mathbb{C}[x, y, z] = \mathbb{C}\langle x, y, z\rangle / \begin{pmatrix} xy - yx \\ xz - zx \\ yz - zy \end{pmatrix}$

1) Skew-polynomial algebra

$A_q = \mathbb{C}_q[x, y, z] = \mathbb{C}\langle x, y, z\rangle / \begin{pmatrix} xy - qyx \\ xz - qzx \\ yz - qzy \end{pmatrix}$ $\quad q \in \mathbb{C}^*$

$\boxed{\text{If } q = \text{root of unity} \Rightarrow \text{finite } / Z(A)}$

2) For $[a:b:c] \in \mathbb{P}^2$:

$A_{(a:b:c)} = \mathbb{C}\langle x, y, z\rangle / \begin{pmatrix} axy + byx + cz^2 \\ axz + bzx + cy^2 \\ ayz + bzy + cx^2 \end{pmatrix}$ ⇝ Sklyanin algebras

⇝ Set $\boxed{\mathrm{coh}(\mathbb{P}^2_{nc}) := \mathrm{qgr}\, A}$

'behaves like' $\mathrm{coh}(\mathbb{P}^2)$, resp. $D^b(\mathbb{P}^2)$

e.g. $D^b(\mathbb{P}^2_{nc}) = \langle \pi A, \pi A(1), \pi A(2) \rangle$

$\pi : \mathrm{gr}\, A \longrightarrow \mathrm{qgr}\, A$

DEFINITION

A noncommutative plane curve of degree d is

$\mathrm{qgr}\left(A/(f)\right)$ for $0 \neq f \in Z(A)_d$

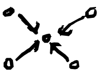
EXAMPLE [Ueyama 2020]

- $A_{q=1}$ ⇝ $f = x^2 + y^2 + z^2$: [Beilinson '78]

  $D^b\left(A_{q=1}/(f)\right) = D^b(\mathbb{P}^1) \cong D^b($rep  $)$

  $f = x^2 + y^2 + z^2 \in Z(A_{q=-1})_2 = \mathbb{C}x^2 \oplus \mathbb{C}y^2 \oplus \mathbb{C}z^2$

- $A_{q=-1}$ ⇝ $f = x^2 + y^2 + z^2 \in Z(A_{q=-1})_2 = \mathbb{C}x^2 \oplus \mathbb{C}y^2 \oplus \mathbb{C}z^2$

  $D^b\left(A_{q=-1}/(f)\right) = D^b($rep  $)$

QUESTIONS

1) $Z(A)_d \neq 0$?

2) What happens for other f ?

II DICTIONARY IN DIM TWO (and ONE)

1) $\operatorname{coh}(X,\mathcal{A})$ $\quad [X = \text{surface}]$

2) $\operatorname{coh}(\mathcal{X})$

3) $\operatorname{qgr}(A)$

Central Proj [Le Bruyn 1995]

A = 3dim AS regular algebra finite/$Z(A)$

Artin '1992 gave an example of a Sklyanin algebra s.t. $\underline{\operatorname{Spec}}_Y Z(\tilde{\mathcal{A}}) \xrightarrow[3:1]{} Y$

$$(Z(A), A) \xrightarrow[\text{Proj}]{} (\underbrace{\operatorname{Proj} Z(A)}_{=Y}, \tilde{\mathcal{A}}) \xrightarrow[\underbrace{Z(\tilde{\mathcal{A}})}_{\text{but finite over } \mathcal{O}_Y} \supseteq \mathcal{O}_Y]{} (\underline{\operatorname{Spec}}_Y Z(\tilde{\mathcal{A}}), \mathcal{A}) =: (X, \mathcal{A})$$

normal Cohen-Macaulay surface

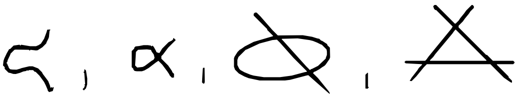
$\Rightarrow \operatorname{qgr} A \simeq \operatorname{coh}(X,\mathcal{A})$

THEOREM [Le Bruyn '95, Mori, Artin, van Gastel 2004] ③ ⟷ ①

If A is 3dim quadratic AS-regular algebra finite/$Z(A)$

$\Rightarrow (X,\mathcal{A}) \cong (\mathbb{P}^2, \mathcal{A})$ with

- $\mathcal{A}$ = maximal $\mathcal{O}_{\mathbb{P}^2}$-order
- $\Delta_{\mathcal{A}}$ = empty,



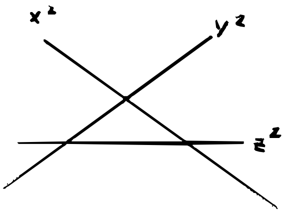
$\overset{\shortparallel}{\{x \in X \mid \mathcal{A} \otimes_X k(x) \text{ not Azumaya}\}}$ $\boxed{\text{RAMIFICATION LOCUS}}$

An $\mathcal{O}_X$-algebra $\mathcal{A}$ is an $\underline{\mathcal{O}_X\text{-order}}$ if
- $\mathcal{A}$ is coherent & torsionfree (as $\mathcal{O}_X$-module)
- $\mathcal{A}_\eta = \mathcal{A} \otimes_X k(X)$ is a central simple $k(X)$-algebra

X integral!

EXAMPLE $A_{q=-1} = k\langle x,y,z\rangle / \begin{pmatrix} xy+yx \\ xz+zx \\ yz+zy \end{pmatrix}$

$\Rightarrow \operatorname{qgr}(A_{q=-1}) \simeq \operatorname{coh}(\mathbb{P}^2, \mathcal{A})$ with $\Delta_{\mathcal{A}} =$



THEOREM [Faber-Ingalls-Okawa-Satriano 2022] ② ⟷ ①

$$\operatorname{coh}(X,\mathcal{A}) \cong \operatorname{coh}(\mathcal{X}, \mathcal{B})$$

smooth DM stack (for $\mathcal{X}$); Azumaya algebra (for $\mathcal{B}$)

- X = coarse space of $\mathcal{X}$
- stacky structure depends on $\Delta_{\mathcal{A}}$

$$\begin{tikzcd}
(X,\mathcal{A}) \arrow[d, "\text{Take root stack in } \Delta_{\mathcal{A}} \text{ [see below]}"] \\
(\mathcal{X}_{\text{root}}, \mathcal{A}_{\text{root}}) \arrow[d, "\text{resolve singularities, take canonical stack}"] \\
(\mathcal{X}, \mathcal{B})
\end{tikzcd}$$

DM stack generally with (linearly reductive quotient) singularities

operation in codim ≥ 2 ⇝ not visible in 1-dim dictionary

smooth DM

THEOREM [Chan-Ingalls 2004]

C = smooth curve

$\mathcal{A}$ = hereditary $\mathcal{O}_C$-order (i.e. gldim $\mathcal{A} = 1$)

at $p \in \Delta_{\mathcal{A}}$: $\mathcal{A} \otimes_C \mathcal{O}_{C,p} \sim \begin{pmatrix} \mathcal{O}_{C,p} & \cdots & \mathcal{O}_{C,p} \\ \mathfrak{m} & \ddots & \vdots \\ \vdots & \ddots & \\ \mathfrak{m} & \cdots\ \mathfrak{m} & \mathcal{O}_{C,p} \end{pmatrix} \subset \operatorname{Mat}_n(\mathcal{O}_{C,p})$

$\boxed{n > 1 \Leftrightarrow p \in \Delta_{\mathcal{A}}}$

$\mathfrak{m} = (r)$

⇝ associated root stack $\sqrt[n]{\operatorname{Spec} \mathcal{O}_{C,p}; p} = \left[\operatorname{Spec}\left(\mathcal{O}_{C,p}[x] / (x^n - r) \right) / \mu_n \right]$

$$⇝ \boxed{\operatorname{coh}(C,\mathcal{A}) \cong \operatorname{coh}\big(\underbrace{\sqrt{C; \Delta_{\mathcal{A}}}}_{\text{smooth root stack}}\big)}$$

III Noncommutative plane curves

THEOREM (B - Belmans - van Garderen)

If $C \subset X$ a curve:

$$\mathrm{coh}(\mathcal{X}\times_X C, \mathcal{B}|_C) \simeq \mathrm{coh}(C, \mathcal{A}|_C) \left(\simeq \mathrm{qgr}\, A/(f)\right)$$

+ properties of $\mathcal{A}|_C$ in terms of $\Delta_{\mathcal{A}} \cap C$

i) $\mathcal{A}|_C$ order $\Leftrightarrow$ $C \not\subset \Delta$

ii) $\mathcal{A}|_C$ Azumaya $\Leftrightarrow$ $C \cap \Delta_{\mathcal{A}} = \emptyset$

iii) $C \pitchfork \Delta_{\mathcal{A}}$ in smooth locus of $\Delta_{\mathcal{A}}$ $\Rightarrow$ $\mathcal{A}|_C$ hereditary

iv) $C \cap \Delta_{\mathcal{A}}$ non-transversally in smooth locus $\Rightarrow$ $\mathcal{A}|_C$ not hereditary.

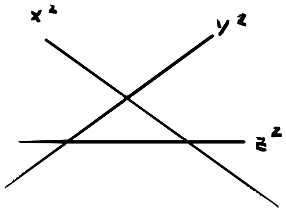
APPLICATION: Noncommutative conics ⇝

studied via Koszul duality

[Ueyama 2020 , Hu-Matsuno-Mori 2021 , Hu - Mori 2023]

$$A_q = \mathbb{C}\langle x,y,z\rangle \Big/ \begin{pmatrix} xy - qyx \\ xz - qzx \\ yz - qzy \end{pmatrix}$$

$$D^b\left(\mathrm{qgr}\, A_q/(f)\right) \simeq \begin{cases} D^b\left(\mathrm{rep}\ \begin{tikzcd} \circ \arrow[r, shift left] \arrow[r, shift right] & \circ \end{tikzcd}\right) & \text{if } q = 1, \\ D^b\left(\mathrm{rep}\ \begin{tikzcd} \circ \arrow[dr] & & \circ \arrow[dl] \\ & \circ & \\ \circ \arrow[ur] & & \circ \arrow[ul] \end{tikzcd}\right) & \text{if } q = -1. \end{cases}$$

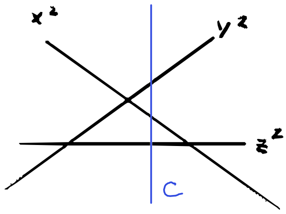


If $q = -1 \rightsquigarrow \mathrm{qgr}(A_q) = \mathrm{coh}(\mathbb{P}^2, \mathcal{A})$ with $\Delta_{\mathcal{A}}$ =

Q (ii): What happens for other f?

① General $f \in \mathbb{C}x^2 + \mathbb{C}y^2 + \mathbb{C}z^2$:

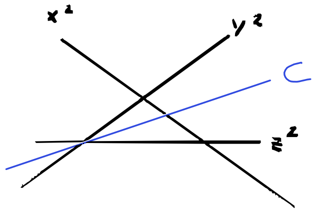
$$\underbrace{D^b(C, \mathcal{A}|_C)}_{\substack{\text{hereditary} \\ \text{order}}} \overset{\text{C'04}}{\simeq} D^b\left(\sqrt[2]{\mathbb{P}^1; 0 + 1 + \infty}\right) \overset{\text{GL'87}}{\simeq} D^b\left(\mathrm{rep}\ \begin{tikzcd} & & \circ \\ \circ \arrow[r, shift left] \arrow[r, shift right] & \circ \arrow[ur] \arrow[r] \arrow[dr] & \circ \\ & & \circ \end{tikzcd}\right) \simeq D^b\left(\begin{tikzcd} \circ \arrow[dr] & & \circ \arrow[dl] \\ & \circ & \\ \circ \arrow[ur] & & \circ \arrow[ul] \end{tikzcd}\right)$$



② $\mathbb{V}(f)$ intersects $\Delta_{\mathcal{A}}$ in singular locus

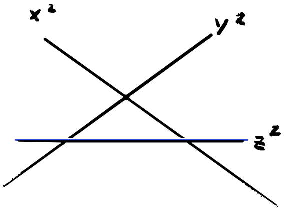
$$\mathrm{coh}(C, \mathcal{A}|_C)$$

↖ Gorenstein order



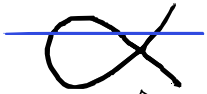
③ $\mathbb{V}(f) \subset \Delta_{\mathcal{A}}$

- $\mathcal{A}|_C$ not an order
- $\mathcal{C}$ = stacky curve with non-trivial gen. stabilizer



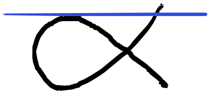
THEOREM [B- Belmans - van Gastel]

In $(\mathbb{P}^2, \mathcal{A})$ = central Proj of a graded Clifford algebra there are 6 Morita equivalence classes of noncommutative conics

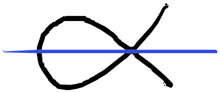


1) generic case

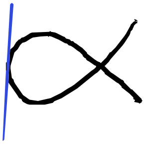
$$A = \mathbb{C}\langle x,y,z\rangle \big/ \begin{pmatrix} xy+yx \\ xz+zx+y^2 \\ yz+zy+x^2 \end{pmatrix}$$



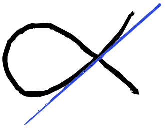
2a) root stack in non-reduced divisor $2p+q$



2b) non-root stack



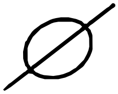
3a) root stack in $3p$



3b) not root stack

Other graded Clifford algebras

$$\rightsquigarrow \quad A = \mathbb{C}\langle x,y,z\rangle \big/ \begin{pmatrix} xy+yx-z^2 \\ xz+zx-y^2 \\ yz+zy-x^2 \end{pmatrix}$$



$$\rightsquigarrow \quad A = \mathbb{C}\langle x,y,z\rangle \big/ \begin{pmatrix} xy+yx \\ xz+zx \\ yz+zy+x^2 \end{pmatrix}$$

PROOF SKETCH

① By classification of terminal orders [Chan-Ingalls 2005] $\mathcal{A}$ is complete locally either

$$\cong \begin{cases} \begin{pmatrix} k[[x,y]] & k[[x,y]] \\ (x) & k[[x,y]] \end{pmatrix} & \text{if } p \in \Delta^{sm} \\[2ex] \dfrac{k[[x,y]]\langle i,j\rangle}{(i^2-x,\, j^2-y,\, ij+ji)} & \text{if } p \in \Delta^{sing} \end{cases}$$

② [Burban-Drozd 2022] $\operatorname{coh}(C,\mathcal{A}) \cong \operatorname{coh}(C',\mathcal{A}') \Longleftrightarrow \exists f: C \longrightarrow C'$ isomorphism of curves s.t.

- $\mathcal{A}_\eta \cong \mathcal{A}'_{f(\eta)}$ of central simple algebras
- $f(\Delta_{\mathcal{A}}) = \Delta_{\mathcal{A}'}$
- $\forall p \in C: \hat{\mathcal{A}}_p \cong \hat{\mathcal{A}}'_{f(p)}$

③ Intersection theory for line & cubic on $\mathbb{P}^2$ + 3-transitivity give 6-iso classes. □

APPLICATION: SKEW CUBICS [Kanazawa '15]

$$A = \mathbb{C}\langle x, y, z\rangle \Big/ \begin{pmatrix} xy - q_1 yx \\ xz - q_2 zx \\ yz - q_3 zy \end{pmatrix} \quad \text{s.t. } q_i^3 = 1$$

$$\rightsquigarrow Z(A)_3 = \mathbb{C}x^3 \oplus \mathbb{C}y^3 \oplus \mathbb{C}z^3$$

PROPOSITION [B- Belmans - van Gaalen]

- If $q_1^{-1} q_2^{-1} = q_1 q_3^{-1} = q_2 q_3$

$$\Rightarrow \forall f \in Z(A)_3 : \operatorname{qgr}\Big(\underbrace{A/(f)}_{\text{1-CY by Kanazawa}}\Big) \simeq \operatorname{coh}(\underset{\substack{\uparrow \\ \text{define by } f \text{ in } \mathbb{P}^2}}{E})$$

- Otherwise

$$\forall f = ax^3 + by^3 + cz^3 \in Z(A)_3 \text{ s.t. } abc \neq 0 : \operatorname{qgr}\Big(A/(f)\Big) \simeq \underbrace{\operatorname{coh}\Big(\sqrt[3]{\mathbb{P}^1; 0 + 1 + \infty}\Big)}_{\text{fractional Calabi-Yau of dim } 3/3}$$

$\Delta_{\cancel{\infty}} =$