# Noncommutative Resolutions of Kleinian Singularities

D.A.N.C.E.

11 May, 2026

1. Recap: the McKay correspondence
2. More resolutions
3. Hilbert schemes and Nakajima quiver varieties
4. Monoidal structures

$\left.\begin{array}{l}\text{2.}\\ \text{3.}\end{array}\right\}$ joint work with Ruth Wye

$\}$ 4. ongoing work with Austin Hubbard

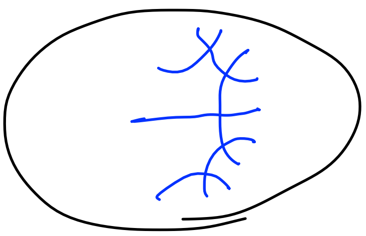
---

**1.** $\Gamma < SL_2(\mathbb{C})$ finite subgroup

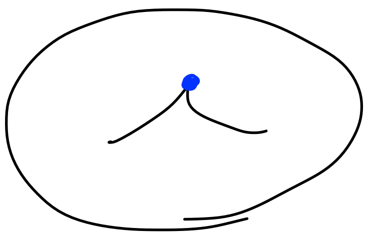
$\rightsquigarrow \Gamma \curvearrowright \mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[x,y])$

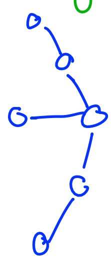
$$\boxed{\mathbb{A}^2/\Gamma = \mathrm{Spec}(\mathbb{C}[x,y]^\Gamma)}$$

Kleinian singularity

$Y$ 

minimal resolution $\Big\downarrow$

$\mathbb{A}^2/\Gamma$ 

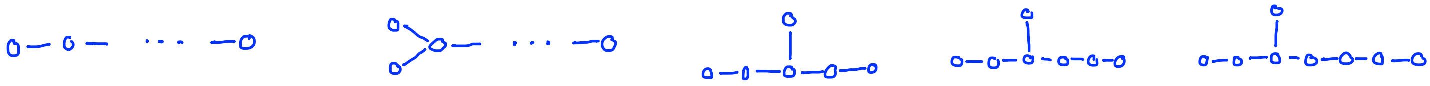


$\left\{\begin{array}{c}\text{ADE}\\ \text{Dynkin}\\ \text{diagrams}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{finite}\\ \Gamma < SL_2(\mathbb{C})\\ \text{up to}\\ \text{conj.}\end{array}\right\}$

$\updownarrow$

$\left\{\begin{array}{c}\text{Kleinian}\\ \text{Sing up}\\ \text{to isom.}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{finite}\\ \Gamma < SL_2(\mathbb{C})\\ \text{up to conj.}\end{array}\right\}$

$A_n \quad (n \geq 1)$ $\qquad$ $D_n \quad (n \geq 4)$ $\qquad$ $E_6$ $\qquad$ $E_7$ $\qquad$ $E_8$



---

$$\begin{tikzcd} Y \arrow[rr, dashed, bend left, leftrightarrow, "\text{McKay correspondence}"] \arrow[dr] & & {[\mathbb{A}^2/\Gamma]} \arrow[dl] \\ & \mathbb{A}^2/\Gamma & \end{tikzcd}$$

$\mathrm{Coh}([\mathbb{A}^2/\Gamma]) = \mathrm{Coh}_\Gamma(\mathbb{A}^2)$

both "crepant" resolutions

<u>Theorem</u> [Kapranov – Vasserot ~2000]

$$\boxed{D^b(\mathrm{Coh}(Y)) \simeq D^b(\mathrm{Coh}([\mathbb{A}^2/\Gamma]))}$$

<u>Conjecture</u> [Bondal, Kawamata] The same is true for any two crepant resolutions of a given Gorenstein singularity.

# Noncommutative algebras

[van den Bergh]

$\exists$ a n.c. algebra $\Pi$ with $Z(\Pi) = \mathbb{C}[x,y]^\Gamma$ s.t.

$$\boxed{\mathrm{mod}(\Pi) \simeq \mathrm{Coh}([\mathbb{A}^2/\Gamma]).}$$

$\Pi$ is a n.c. resolution [van den Bergh] of $\mathbb{A}^2/\Gamma$.

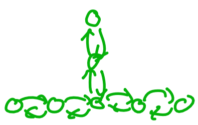
$$\boxed{\begin{array}{ccc} D^b(\mathrm{Coh}(Y)) & \xrightarrow[\mathrm{RHom}(V^*,-)]{\sim} & D^b(\mathrm{mod}(\Pi)) \\ & & \wr\wr \\ & & D^b(\mathrm{Coh}([\mathbb{A}^2/\Gamma])) \end{array}}$$

where $V$ is a v.b. on $Y$ with $\mathrm{End}(V) \cong \Pi$.

$\Pi$ is usually either

$[\mathbb{A}^2/\Gamma]$

Preprojective algebra



Skew-group algebra

$\mathbb{C}[x,y] \otimes \mathbb{C}\Gamma$

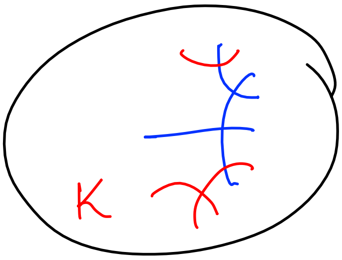
$$\begin{array}{ccccc} g \cdot & f & = & g(f) \cdot g \\ \cap & \cap & \nearrow & \\ \Gamma & \mathbb{C}[x,y] & & \end{array}$$

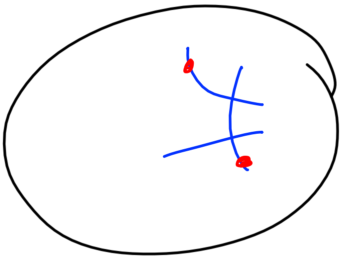
$(f \otimes g)(f' \otimes g')$

$$Y = \Gamma\text{-}\mathrm{Hilb}(\mathbb{A}^2) = \mathrm{Hilb}^1([\mathbb{A}^2/\Gamma])$$

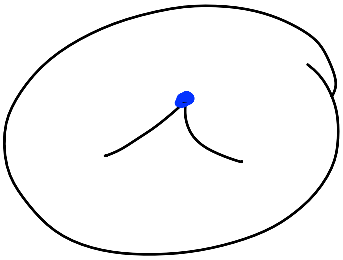
$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ {[\mathbb{A}^2/\Gamma]} & & Y \end{tikzcd}$$

## 2.





crepant partial resolution



$$\begin{tikzcd}
Y \arrow[dr, "h"'] & \mathcal{Y}_K \arrow[d] & {[\mathbb{A}^2/\Gamma]} \arrow[ddl] \\
 & Y_K \arrow[d, "g"'] & \\
 & \mathbb{A}^2/\Gamma &
\end{tikzcd}$$

**Theorem** [Chen–Tseng]

$$\boxed{D^b(\mathrm{Coh}(Y)) \simeq D^b(\mathrm{Coh}(\mathcal{Y}_K)) \simeq D^b(\mathrm{Coh}([\mathbb{A}^2/\Gamma]))}$$

**Theorem** [B-Wye] Set $\mathcal{E}_K := h_* \mathcal{E}nd(V)$, then $\boxed{\mathrm{mod}(\mathcal{E}_K) \simeq \mathrm{Coh}(\mathcal{Y}_K)}$ (over $Y_K$).

$$\begin{tikzcd}
D^b(\mathrm{Coh}(Y)) \arrow[r, "\sim"] \arrow[d, "\wr\ \text{Morita}", "V\otimes -"'] & D^b(\mathrm{Coh}(\mathcal{Y}_K)) \arrow[r, "\sim"] \arrow[d, "\wr\ \text{Morita}"] & D^b(\mathrm{Coh}([\mathbb{A}^2/\Gamma])) \arrow[d, "\wr\ \text{Morita}"] \\
D^b(\mathrm{mod}(\mathcal{E}nd(V))) \arrow[r, "\sim", "Rh_*"'] & D^b(\mathrm{mod}(\mathcal{E}_K)) \arrow[r, "\sim", "Rg_*"'] & D^b(\mathrm{mod}(\Pi))
\end{tikzcd}$$

$Y$ $\qquad$ $Y_K$ $\qquad$ $\mathbb{A}^2/\Gamma$ $\qquad$ [van den Bergh]

# 3.

Smooth, $2n$-dimensional, birational

Hilbert schemes of points: fine moduli spaces of $[\mathcal{O} \twoheadrightarrow F]$ in Coh, where $F$ has zero-dim. support and $[F] = n[pt]$ in $K_{cpt}$

$$\mathrm{Hilb}^n(Y) \cong \mathcal{M}^{n\delta}_{\zeta_-}(\Pi)$$

[Kuznetsov '07, Nakajima, Haiman]

$$\mathrm{Hilb}^n(\mathcal{Y}_K) \cong \mathcal{M}^{n\delta}_{\zeta_K}(\Pi)$$

[B-Wye]

$$\mathrm{Hilb}^n([\mathbb{A}^2/T]) \cong \mathcal{M}^{n\delta}_{\zeta_+}(\Pi)$$

$$\| $$

$$nT\text{-}\mathrm{Hilb}(\mathbb{A}^2)$$

Nakajima quiver varieties: GIT quotients parametrising framed, $\zeta$-stable $\Pi$-modules of dimension vector $n\delta$

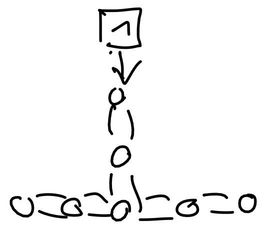
$\zeta \in \mathbb{Z}^I$

$I = \mathrm{Irrep}(T)$

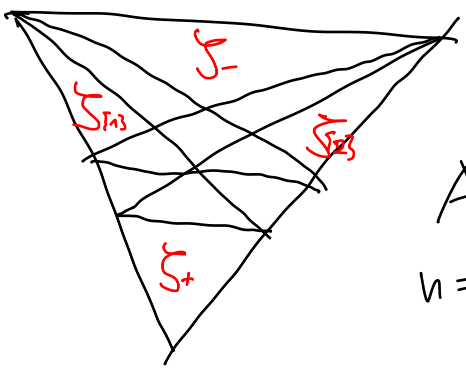
$\delta \in \mathbb{N}^I$

$\delta_i = \dim(\rho_i)$

Variation of t-structures $\longleftrightarrow$ Variation of stability (VGIT)




[Bellamy-Craw]




$A_2$, $n=3$

4.

Theorem [Fukuyama-Iwanari]

A "nice" DM-stack $\mathcal{Y}$ can be recovered from $(D^b(\mathcal{Y}), \otimes)$ as $\{D^b(\mathcal{Y}) \xrightarrow{\text{mon.}} D^b(pt)\}$.

| | $K_0(Y)\otimes\mathbb{C}$ | $K_0(\mathcal{Y}_K)\otimes\mathbb{C}$ | $K_0([\mathbb{A}^2/\Gamma])\otimes\mathbb{C}$ |
|---|---|---|---|
| Spec |  | | |

Q: What would be a natural parameter space for this deformation?

Relation with quantum cohomology/K-theory?

Q: How does one "deform" monoidal structures on a derived category?