Categorical resolutions

(an optimistic view)

Calum Crossley

3 Nov 2025

DK-Conjecture (Bondal–Orlov '02, Kawamata '02)

Take a roof
$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$
of smooth projective varieties.

- If $f^*K_X = g^*K_Y$, then $D^b(X) = D^b(Y)$.
- If $f^*K_X > g^*K_Y$, then $D^b(X) = \langle D^b(Y), \ldots \rangle$.

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Example (Crepant resolutions should be *D*-unique)

All crepant ($K_{\widetilde{X}} = \pi^* K_X$) resolutions $\pi : \widetilde{X} \to X$ of a Gorenstein singularity are K-equivalent $\implies$ conjecturally D-equivalent.

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$\Longrightarrow$ At the level of $D^b(-)$, MMP stops at a *smooth* minimal model. (when a crepant resolution exists)

Minimal Resolution Conjecture (Bondal–Orlov '02)

- $D^b(X)$ has a minimal "nc-desingularization", which embeds fully faithfully (SOD) in all other desingularizations, and hence is unique.
- If $\pi : \widetilde{X} \to X$ is a crepant resolution, then $D^b(\widetilde{X})$ is the minimal desingularization of $D^b(X)$.

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Remark ("stacky crepant resolutions")

By [BKR], crepant resolutions of certain (cDV) quotient singularities are D-equivalent to the (smooth) orbifold quotient $[X/G]$.

Generalizing this, $D^b([X/G])$ should always be the minimal desingularization for the coarse moduli space X/G.

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Remark

Even with a good definition, there is no clear proof approach; how do you construct a functor between arbitrary desingularizations?

Other context

Why else do we care?

- Appears naturally in homological mirror symmetry (e.g. degenerations, or proving mirror symmetry for singularities via the resolution)
- Appears naturally in homological projective duality (e.g. K3 constructed from a singular cubic fourfold giving a categorical resolution of the Kuznetsov component).
- (Conjectural) unique minimal resolutions lead to (conjectural) invariants of singularities (like dual graphs of ADE surfaces).
- Like *DK*-conjecture, predicted equivalences between minimal resolutions can be *unexpected*, hinting at deeper structure in the constructions used to build the categories involved.
- There are actual hands-on constructions, allowing us to study concrete examples, and we find interesting structure which is poorly understood (e.g. null categories, relative singularity categories).

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Remark

- Crepant resolution $\implies$ rational singularities.
- Terminal singularities $\implies$ rational singularities.

Categorical resolutions

Definition (Kuznetsov ’08)

A *weak categorical resolution* is a smooth category $\mathscr{C}$ with adjoint functors

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The justification for this definition when X has irrational singularities is a theorem of Kuznetsov and Lunts, which we will see in a moment.

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Example

Take $X = \operatorname{Spec} k[x]/x^2$. Then $D^b(X) = \langle k[x]/x \rangle$, and $\operatorname{Ext}^*_X(k,k) = k[\theta]$ where $|\theta| = 1$, so by Koszul duality $D^b(X) = D^b(k[\theta])$ which is smooth like $\mathbb{A}^1$.

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Categorical resolutions

Definition

A *proper categorical resolution* is a weak categorical resolution such that

- $\mathscr{C}$ is "linear over $D^b(X)$", i.e. $\mathrm{Hom}_{\mathscr{C}} = R\Gamma_X \circ \mathcal{H}om_{\mathscr{C}}$ where $\mathcal{H}om_{\mathscr{C}}(-,-) \in D^b(X)$, plus naturality conditions.
- The functors π^*, π_* respect this structure.

Jargon: a module over the tensor category $(\mathfrak{Perf}(X), \otimes)$.

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Idea: $\mathcal{H}om_{\mathscr{C}}(-,-) \in D^b(X)$ means Homs in $\mathscr{C}$ are finite rank (coherent) relative to X $\implies$ proper category relative to X.

Definition (Kuznetsov '08)

A *relative Serre functor* for a proper categorical resolution is an endofunctor $S : \mathscr{C} \to \mathscr{C}$ with a natural isomorphism

$$\mathcal{H}om_{\mathscr{C}}(x, S(y)) = \mathcal{H}om_X(\mathcal{H}om_{\mathscr{C}}(y,x), \mathcal{O}_X).$$

The resolution is *strongly crepant* if $\mathrm{id}_{\mathscr{C}}$ is a relative Serre functor.

Categorical resolutions

Remark

It follows that $\pi^! = S \circ \pi^*$, so strongly crepant $\implies$ weakly crepant.

Example

For $\pi : \widetilde{X} \to X$ the relative Serre functor is $- \otimes \omega_{\widetilde{X}/X}[\dim \widetilde{X} - \dim X]$, and strong crepancy $\iff$ weak crepancy $\iff$ crepancy: $\omega_{\widetilde{X}/X} = \mathcal{O}_{\widetilde{X}}$.

Theorem (Kuznetsov–Lunts '15)

Proper categorical resolutions exist for any separated scheme of finite type in characteristic zero.

Conjecture (Kuznetsov '08)

Strongly crepant categorical resolutions are minimal.

Example: dual numbers

Take $X = \operatorname{Spec} k[x]/x^2$ again. A proper categorical resolution has to be a smooth and proper category (since $k[x]/x^2$ is finite over k).

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Use the compactification $\mathbb{P}^1_{\theta:\phi}$, $|\theta| = 1$ of $D^b(X) = D^b(k[\theta])$. Beilinson:

$$\mathscr{C} = D^b\Big(\begin{tikzcd} \bullet \arrow[r, shift left, dashed, rightsquigarrow, "\theta"] \arrow[r, shift right, "\phi"'] & \bullet \end{tikzcd} \Big),$$

which contains $\mathfrak{Perf}(X)$ as the subcategory generated by $\mathcal{O}_{[0:1]} = \operatorname{cone}(\theta)$.

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Two perspectives:

- Smoothing $\mathfrak{Perf}(X)$: finding a smooth category with an object having endomorphisms $k[x]/x^2$ (skyscraper sheaf in $\mathbb{P}^1$).
- Compactifying $D^b(X)$: compactifying $\mathbb{A}^1$ to $\mathbb{P}^1$.

Minimality, SOD's and null categories

Suppose $\overline{\mathscr{C}}$ is the minimal resolution:

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with $\mathscr{C} = \langle \overline{\mathscr{C}}, \mathcal{A} \rangle$.

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$$\begin{aligned} &\{\text{categorical resolutions of } D^b(X)\} \\ &\qquad = \{\text{categorical resolutions of the absorbing subcategory } \mathcal{X}\}. \end{aligned}$$

Note: if $\mathcal{X} = D^b(R)$, we get $D_{\mathrm{Sg}}(X) = D_{\mathrm{Sg}}(R)$.

Constructions

Auslander algebras (Kuznetsov–Lunts)

If S is a non-reduced thickening of a smooth variety S_0, there is a sheaf of algebras $\mathcal{A}_{S/S_0}$ on S which is a proper categorical resolution, with an SOD:

$$D^b(\mathcal{A}_{S/S_0}) = \langle D^b(S_0), \dots, D^b(S_0) \rangle.$$

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Example

The dual numbers $k[x]/x^2$ are a 2nd order thickening of a point, so we get an SOD: $D^b(\mathcal{A}) = \langle D^b(\mathrm{pt}), D^b(\mathrm{pt})\rangle$. The Auslander resolution $D^b(\mathcal{A})$ is equivalent to the $\mathbb{P}^1_{\theta:\phi}$ resolution.

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The dual numbers $k[x]/x^2$ are a 2nd order thickening of a point, so we get an SOD: $D^b(\mathcal{A}) = \langle D^b(\mathrm{pt}), D^b(\mathrm{pt}) \rangle$. The Auslander resolution $D^b(\mathcal{A})$ is equivalent to the $\mathbb{P}^1_{\theta:\phi}$ resolution.

The Kuznetsov–Lunts existence theorem is proved by gluing Auslander resolutions of thickenings of the blowup centres appearing in a resolution of singularities.

Constructions

NC(C)R's (Van den Bergh)

In nice situations (e.g. quotient singularities), when $X = \operatorname{Spec} R$, there can be a reflexive R-module M such that $\Lambda = \operatorname{End}_R(M)$ has good homological properties.

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Caveat

NCCR's are not known to always give categorical resolutions due to a technical obstruction, but it is expected and known in many cases.

Matrix factorizations

Definition

A *Landau–Ginzburg model* is a smooth variety U with a $\mathbb{C}^*$-action (*R-charge*) and a global function $W \in H^0(U, \mathcal{O}_U)$ (the *superpotential*) with is $\mathbb{C}^*$-equivariant of weight 2.

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Fact

Because $W = \frac{1}{2}dd + \frac{1}{2}dd$ and $\partial_x(W) = \partial_x(d)d + d\partial_x(d)$ are null-homotopies, the category $\mathrm{MF}(U, W)$ is linear over $\mathrm{Crit}(W) \subset U$.

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Example (Classical Knörrer periodicity)

$\mathcal{E} = \left(\mathcal{O} \underset{y}{\overset{x}{\rightleftarrows}} \mathcal{O}[-1] \right) \in \mathrm{MF}(\mathbb{A}^2, xy)$ is an exceptional object, where R-charge is $|x| = 0$, $|y| = 2$. It is quasi-isomorphic to $\mathcal{O}/y$, so

$$\mathcal{H}om(\mathcal{E}, \mathcal{E}) = \mathcal{H}om\left(\mathcal{O} \underset{y}{\overset{x}{\rightleftarrows}} \mathcal{O}[-1] , \mathcal{O}/y \right)$$

$$= (\mathcal{O}/y[1] \xrightarrow{x} \mathcal{O}/y) = \mathcal{O}/(x, y).$$

This is a sheaf on $\mathrm{Crit}(xy) = \{x = y = 0\}$, as expected.

Matrix factorizations

Theorem (Knörrer periodicity (Orlov '06, ...))

Suppose $X \subset U$ is a hypersurface cut out by $f \in H^0(U, \mathcal{L})$. Then

$$D^b(X) \simeq \mathrm{MF}(\mathrm{Tot}\,\mathcal{L}^\vee, fp); \qquad \mathscr{F} \mapsto \left(\ \mathscr{F} \underset{p}{\overset{f}{\rightleftarrows}} \mathscr{F} \otimes \mathcal{L}[-1] \ \right),$$

where the fiber coordinate $p \in H^0(\mathrm{Tot}\,\mathcal{L}^\vee, \mathcal{L}^\vee)$ is given R-charge $|p| = 2$.

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Here $\mathrm{Crit}(fp) = \{f = p = 0\} \cup \{\mathrm{Crit}(f)\} = X \cup \mathcal{L}^\vee|_{\mathrm{Sing}(X)}$.

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Idea ("Exoflops" Aspinwall '14)

We find proper categorical resolutions of X by compactifying $\mathrm{Crit}(fp)$ inside $\mathrm{Tot}\,\mathcal{L}^\vee$, i.e. partially compactifying $\mathrm{Tot}\,\mathcal{L}^\vee$.

Remark

What does $\mathfrak{Perf}(X)$ correspond to? Vector bundles $\mathcal{E}$ map to

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Remark

By Knörrer periodicity $f \rightsquigarrow f + x^2 + y^2$, studying curves / surfaces also gives results for higher-dimensional hypersurfaces.

Example A_1

Node $X = \{xy = 0\} \subset \mathbb{A}^2$, normalization $\widetilde{X} \subset \mathrm{Bl}_0\, \mathbb{A}^2 = \mathrm{Tot}\, \mathcal{O}(-1)_{\mathbb{P}^1}$.

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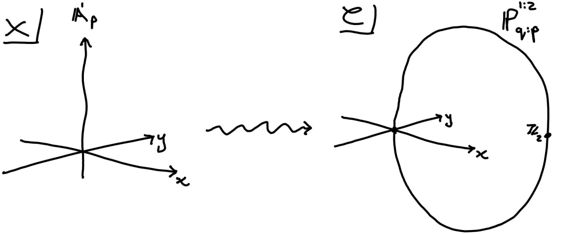

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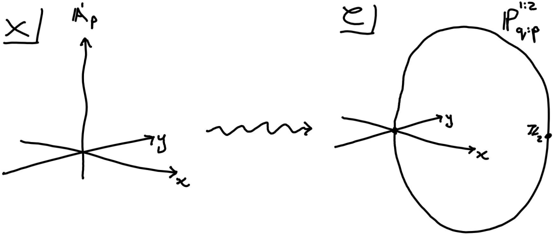

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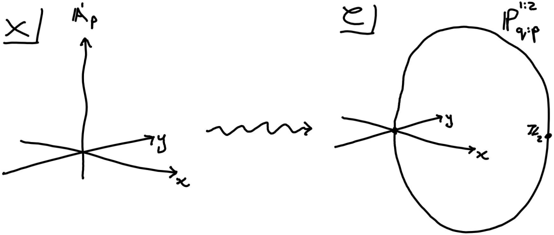

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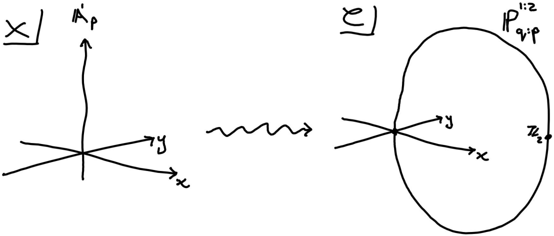

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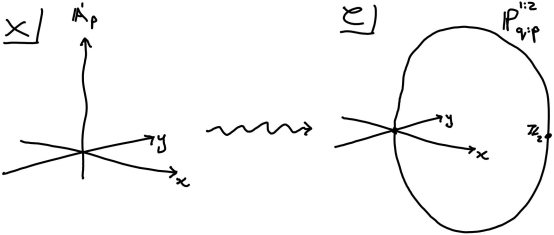

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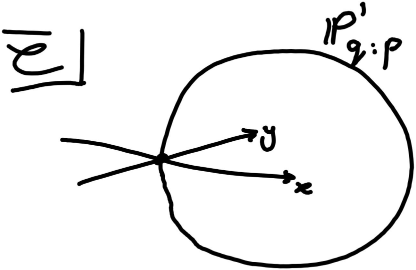

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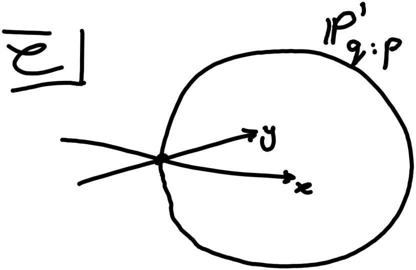

The null category at $\{q = 0\}$ is $D^b_0(\mathbb{A}^1) = \mathfrak{Perf}(k[\theta]/\theta^2)$, via Koszul duality (work out R-charge $\implies |\theta| = 3 \implies$ 3-spherical object).

Remark

By localization, $D_{\mathrm{Sg}}(X) = \mathsf{MF}_{\{p \neq 0\}}(\operatorname{Tot} \mathcal{L}^\vee, fp)$ is the branch bit we are seeing, separated from X, pre-compactification.

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- The data of $\mathscr{C}/\mathfrak{Perf}(X) \to D_{\mathrm{Sg}}(X)$ should then determine the resolution via gluing with $D^b(X) \to D_{\mathrm{Sg}}(X)$.
- In particular, if $D_{\mathrm{Sg}}(X') = D_{\mathrm{Sg}}(X)$ then X' and X should have the same categorical resolution theory.

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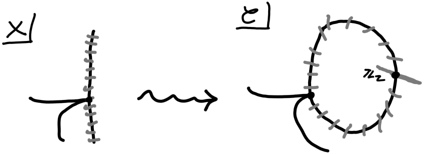

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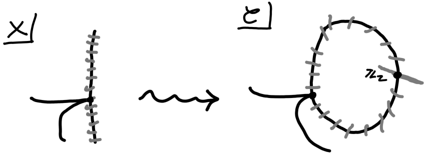

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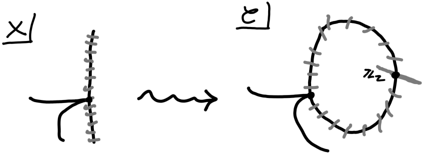

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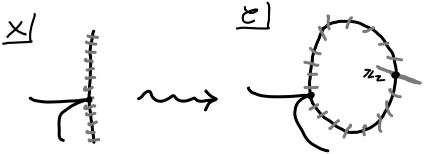
$$\mathscr{C} = \mathsf{MF}(\mathrm{Tot}\,\mathcal{O}(-1)^2_{\mathbb{P}_{1:2}}, (y^2 - x^3 q)p) = \langle D^b(\widetilde{X}), D^b(\mathrm{pt})\rangle.$$



Since $\mathrm{Crit}(y^2 - x^3 q) = \{y = x^3 = x^2 q = 0\}$, we get more non-reduced fuzz at $q = 0$.

$$\mathscr{C}\,/\,\mathfrak{Perf}(X) = \mathsf{MF}([\mathbb{A}^3/\mathbb{Z}_2], y^2 - x^3 q) = \mathsf{MF}(\mathbb{A}^2, x^3 q) = D^b(k[x]/x^3),$$

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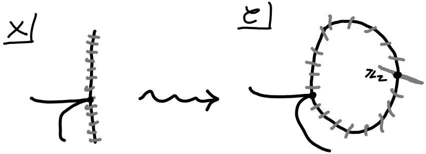

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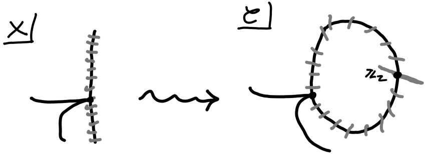

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and the null category at $\{q = 0\}$ is $\mathfrak{Perf}(\mathbb{C}[x]/x^3)$; we find an object with endomorphisms $k[x]/x^3$ (work out R-charge $\implies |x| = 1$).

Example $k[x]/x^n$

We can resolve $k[x]/x^n$ by compactifying $(\mathbb{A}^2, x^n p)$:

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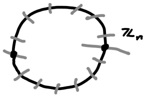

$$\mathscr{C}/\mathfrak{Perf}(X) = \mathrm{MF}([\mathbb{A}^2/\mathbb{Z}_n], x^n) = \langle D^b(\mathbb{A}^1), \ldots, D^b(\mathbb{A}^1)\rangle$$

with the null category at $q = 0$ corresponding to

$$\ker \pi_* = \langle D^b_0(\mathbb{A}^1), \ldots, D^b_0(\mathbb{A}^1)\rangle$$

i.e. a semi-orthogonal sequence of spherical objects.

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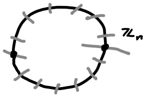

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Thank you!