Categorical Torelli theorems for cyclic covers

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Q1 How do geometric invariants behave under finite group actions?

§1 Categorical Torelli problems

(Nice survey: Pertusi–Stellari '22)

X, X' sm. proj. varieties $/\mathbb{C}$ — mostly true for char = 0 or large enough

Th'm [Gabriel '62] $\mathrm{Coh}(X) \simeq \mathrm{Coh}(X')$ (as $\mathbb{K}$-lin. cats) $\Rightarrow X \simeq X'$ (as $\mathbb{K}$-varieties) invariant!

Th'm [Bondal–Orlov '01] K_X or $-K_X$ ample, $D^b(X) \simeq D^b(X') \Rightarrow X \simeq X'$ huge!

Q: What about "less information"

Def'n $\mathcal{D}$: triangulated category. A semiorthogonal decomposition (S.O.D.) is $\mathcal{D} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ s.t.

1. $\mathcal{A}_i$ full Δ-ed subcategories — "inclusion is full (surj. on homs)"
2. (S.O.) $\mathrm{Hom}_{\mathcal{D}}(\mathcal{A}_i, \mathcal{A}_j) = 0$ if $i > j$
3. (D.) $\forall\, E \in \mathcal{D}$, $\exists$ $0 \to E_n \to \dots \to E_1 \to E_0 = E$ s.t. $\mathrm{cone}(E_i \to E_{i-1}) \in \mathcal{A}_i$ (actually unique!)

Fano varieties have non-trivial SODs!
($\langle \mathcal{O}_X, \dots, \mathcal{O}_X((i-1)H) \rangle$ exc. coll'n)

Ex $X = \mathbb{P}^n$, $D^b(X) = \langle \mathcal{O}_X, \dots, \mathcal{O}_X(n) \rangle$ "$\mathcal{O}_X = \langle \mathcal{O}_X \rangle$"

Ex $X \subset \mathbb{P}^4$ cubic 3fold, $D^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1) \rangle$

where $\mathcal{A}_X := \{ E \in D^b(X) : \mathrm{Hom}^\bullet(\mathcal{O}_X, E) = \mathrm{Hom}^\bullet(\mathcal{O}_X(1), E) = 0 \}$

"Kuznetsov component" ← ad hoc!

Th'm [Bernardara–Macrì–Mehrotra–Stellari '12] X, X' cubic 3folds.

$\mathcal{A}_X \simeq \mathcal{A}_{X'} \Rightarrow X \simeq X'$ — proof uses Bridgeland moduli spaces

Does this happen for other Fanos?

Expectation: $\mathcal{A}_X$ contains "all essential information about X"

Categorical Torelli problem: X, X' Fanos of same defo. type with Kuznetsov components $\mathcal{A}_X \simeq \mathcal{A}_{X'} \Rightarrow X \simeq X'$?

There are 17 families of smooth Fano threefolds X with $\operatorname{Pic} X = \mathbf{Z} = \langle H \rangle$. They are classified by their *index* i s.t. $K_X = -iH$, and *degree* $d = H^3$.

$i = 1,\ d = 2g_X - 2$			
g_X	$\mathrm{D^b}(X_d)$	CTT?	Refined CTT?
12	$\langle \mathcal{E}_4, \mathcal{E}_3, \mathcal{E}_2, \mathcal{O} \rangle$	no (birational)	yes
10	$\langle \mathrm{D^b}(C_2), \mathcal{E}_2, \mathcal{O} \rangle$	no (birational)	yes
9	$\langle \mathrm{D^b}(C_3), \mathcal{E}_3, \mathcal{O} \rangle$	no (birational)	yes
8	$\langle \mathcal{A}_{X_{14}}, \mathcal{E}_2, \mathcal{O} \rangle$	no (birational)	yes
7	$\langle \mathrm{D^b}(C_7), \mathcal{E}_5, \mathcal{O} \rangle$	yes	
6	$\langle \mathcal{A}_{X_{10}}, \mathcal{E}_2, \mathcal{O} \rangle$	no* (birational)	yes
5	$\langle \mathcal{A}_{X_8}, \mathcal{O} \rangle$	yes (rigid)	
4	$\langle \mathcal{A}_{X_6}, \mathcal{O} \rangle$	???**	
3	$\langle \mathcal{A}_{X_4}, \mathcal{O} \rangle$	???	
2	$\langle \mathcal{A}_{X_2}, \mathcal{O} \rangle$	[DJR, LZ]	

resisted old techniques

$i = 2$		
d	$\mathrm{D^b}(Y_d)$	CTT?
5	$\langle \mathcal{F}_3, \mathcal{F}_2, \mathcal{O}, \mathcal{O}(1) \rangle$	yes
4	$\langle \mathrm{D^b}(C_2), \mathcal{O}, \mathcal{O}(1) \rangle$	yes
3	$\langle \mathcal{A}_{Y_3}, \mathcal{O}, \mathcal{O}(1) \rangle$	yes
2	$\langle \mathcal{A}_{Y_2}, \mathcal{O}, \mathcal{O}(1) \rangle$	yes
1	$\langle \mathcal{A}_{Y_1}, \mathcal{O}, \mathcal{O}(1) \rangle$	[DJR, LRZ]

i	$\mathrm{D^b}(X)$	CTT?
3	$\langle S, \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(2) \rangle$	rigid
4	$\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3) \rangle$	rigid

Notation: C_g is a smooth curve of genus g.
$\mathcal{E}_i$, $\mathcal{F}_j$, S are vector bundles.
*: for ordinary, yes for special.
**: yes for quartic hypersurfaces.

⚠ X_2 and Y_1 arise as branched cyclic covers.
Let $\mathcal{X}_2$ and $\mathcal{Y}_1$ denote their moduli.

[IDEA: exploit extra structure from cyclic group action]

Thm 1 [DJR]
(1) $X, X' \in \mathcal{X}_2$, X very general, $\Phi : \mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_{X'}$ Fourier–Mukai $\Rightarrow X \simeq X'$
(2) $X, X' \in \mathcal{Y}_1$, ——— $\|$ ———
and Φ commutes with the covering involution $\Rightarrow X \simeq X'$

Rmk
(1) proven by different methods in [Xun Lin – Shizhuo Zhang '23]
(2) proven without involution assumption in [Lin-Rennemo-Zhang '24]

§ 2 Branched cyclic covers

SET UP: $\mu_n \circlearrowright X$: degree n cyclic cover ramified over Z

$$\begin{tikzcd} & X \arrow[d, "f", "n:1"'] \\ Z \arrow[ur, "j"] \arrow[r, hook, "i"] & Y \end{tikzcd}$$

Y : algebraic variety or proper DM stack (· alg stack, étale cov. by scheme · locally $[X/G]$ G finite)

$Z \in |\mathcal{O}_Y(nd)|$

Assume
① Y has a rectangular Lefschetz decomposition
i.e. $\exists\ \mathcal{O}_Y(1)$ and admissible $\mathcal{B} \subset \mathrm{D^b}(Y)$ s.t.

$$\mathrm{D^b}(Y) = \langle \mathcal{B}, \mathcal{B}(1), \dots, \mathcal{B}(m-1) \rangle$$

e.g. (weighted) projective space, Grassmannians, other homogeneous spaces, ...

② $M := m - (n-1)d > 0$

$\underline{\text{Th}^{m}}$ [Kuznetsov–Perry '17]: $D^b(X) = \langle \underbrace{\mathcal{A}_X}_{\text{Kuznetsov component}}, f^*B, \dots, f^*B(M-1) \rangle$ fully faithful

(Q2)

$$\begin{tikzcd} & X' \arrow[d, "n:1"] \\ Z' \arrow[ur] \arrow[r, hook] & Y \end{tikzcd}$$

$Z' \in |\mathcal{O}(nd)|$

$\mathcal{A}_X \simeq \mathcal{A}_{X'} \overset{?}{\Rightarrow} Z \simeq Z' \overset{?}{\Rightarrow} X \simeq X'$

<u>KEY OBSERVATION</u>: $D^b(X)$ doesn't "see" Z, but $D^b([X/\mu_n]) \simeq D^b(X)^{\mu_n}$ does!

to make more precise, need:

<u>INTERLUDE</u>: Equivariant categories

G: finite group, $\mathcal{D}$: [more gen: $\mathcal{C}$ pre-add. lin / ring k, (char k, $|G|$) $= 1$]

<u>Def^n</u> A (strict) (right) action of G on $\mathcal{D}$ is given by

1. autoequivalences $\phi_g : \mathcal{D} \xrightarrow{\sim} \mathcal{D}$ $\forall g \in G$
2. isomorphisms $\varepsilon_{g,h} : \phi_g \circ \phi_h \xrightarrow{\sim} \phi_{hg}$ $\forall g,h$
3. "associativity"

$\Rightarrow$ more than $G \leq \mathrm{Aut}(\mathcal{D})$

+ some assumps to guarantee Δ-ed

$G \curvearrowright \mathcal{D} \rightsquigarrow \mathcal{D}^G$: G-equivariant category of $\mathcal{D}$ < <u>think</u>: G-equiv. sheaves.

<u>objects</u>: $(E, \{\lambda_g\}_{g\in G})$ s.t. $\lambda_g : E \xrightarrow{\sim} \phi_g(E)$ + compatibility $\boxed{\text{more than "invariants"}}$

- <u>morphs</u>: morphisms in $\mathcal{D}$ that commute w/ λ_g.

$G = \mu_n \curvearrowright D^b(X)$ <u>by pullback</u> and preserves f^*B and hence $\mathcal{A}_X$

$\rightsquigarrow$ SOD for $D^b(X)^{\mu_n} \overset{\text{SOD}}{\supset} \mathcal{A}_X^{\mu_n}$ "equiv. Kuz. comp."

[KP17: another SOD, mutate & compare $\Rightarrow$ describe $\mathcal{A}_X^{\mu_n}$]

To ease notation let $n=2$.

<u>Th^{m}2</u> [DJR] $0 < M < d \Rightarrow \mathcal{A}_X^{\mu_2} = \langle j_* D^b(Z), \underbrace{\varepsilon_1, \dots, \varepsilon_{\underbrace{d-M}_{>0}}}_{=:\mathcal{A}_Z} \rangle$, $\boxed{\begin{array}{l} = B_X'[M-d,-1] \\ \quad \| \\ \langle B_X(M-d)\otimes\rho_1, \dots, \\ \quad B_X(-1)\otimes\rho_1 \rangle \end{array}}$

s.t. ε_i <u>exceptional</u> $(\mathrm{Hom}^\bullet(\varepsilon_i, \varepsilon_i) = \mathbb{C}[0])$

<u>Remark</u> • For $n > 2$, $\mathcal{A}_X^{\mu_n}$ consists of $n-1$ copies of $\mathcal{A}_Z$

$\left[\begin{array}{l} \Phi_k = \amalg_{B_X^{[0,k]}}([0,M-1]) \; (-\otimes\rho_k) \\ \mathcal{A}_X^{\mu_n} = \langle \Phi_0(\mathcal{A}_Z), \Phi_1(\mathcal{A}_Z), \dots, \Phi_{n-2}(\mathcal{A}_Z) \rangle \end{array} \right]$

- This extends [KP] to Z canonically polarised
  - ↳ In their case: $(M = d \Rightarrow \deg K_Z = 0)$ $D^b(Z) \simeq \mathcal{A}_Z^{KP}$, or
    $(M > d \Rightarrow Z$ Fano$)$ $D^b(Z) \supsetneq \mathcal{A}_Z^{KP}$
  - ↳ For us: $\deg K_Z > 0$

- [Orlov] + [Hirano–Ouchi]: analogous result for matrix factorisations

NOW BACK TO Q2 :

$$A_X \simeq A_{X'} \overset{?}{\Rightarrow} Z \simeq Z' \overset{?\,C}{\Rightarrow} X \simeq X'$$

A ? $\searrow$ $A_X^{\mu_n} \simeq A_{X'}^{\mu_n}$ $\nearrow$? B

RUNNING EXAMPLE : X_2

$$\begin{tikzcd} & \mu_2 \curvearrowright X_2 \arrow[d, "f"', "2:1"] \\ Z \arrow[r, hook] \arrow[ur, "j"] & \mathbb{P}^3 \end{tikzcd}$$

$Z \in |\mathcal{O}_{\mathbb{P}^3}(6)|$

$n=2, d=3, m=4$

- $D^b(Y) = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3) \rangle$
- $M = 1 > 0$
- $D^b(X) = \langle A_X, \mathcal{O} \rangle$
- $D^b(X)^{\mu_2} = \langle A_X^{\mu_2}, \langle \mathcal{O} \rangle^{\mu_2} \rangle$ $= \langle f^* D^b(Y) \otimes \rho_0, j_* D^b(Z) \otimes \rho_0 \rangle$

$$A_X^{\mu_2} \overset{\text{Th}^m 2}{=} \langle j_* D^b(Z), \mathcal{E}_1, \mathcal{E}_2 \rangle$$

$\mathcal{E}_1 = \mathcal{O}_X(-2) \otimes \rho_1$
$\mathcal{E}_2 = \mathcal{O}_X(-1) \otimes \rho_1$

Key to B :

Th^m 3 [DJR] • $n=2$, $0<M$, X, X' $\underbrace{\text{prime}}_{\text{Pic rk 1}}$ Fano threefolds, Y weighted projective space

$$\begin{tikzcd} & X \arrow[dr, "2:1"] & & X' \arrow[dl, "2:1"'] & \\ Z \arrow[ur] \arrow[rr, hook] & & Y & & Z' \arrow[ll, hook'] \arrow[ul] \end{tikzcd}$$

$Z \in |\mathcal{O}(2d)|$, $Z' \in |\mathcal{O}(2d)|$

• $\Phi^{\mu_2} : A_X^{\mu_2} \xrightarrow{\sim} A_{X'}^{\mu_2}$ Fourier-Mukai

Then X very general $\Rightarrow$ $H^2_{prim}(Z, \mathbb{Z}) \overset{\text{Hodge isometry}}{\simeq} H^2_{prim}(Z', \mathbb{Z})$

$H^2_{prim}(Z,\mathbb{Z}) = \ker(-\cup h : H^2(Z, \mathbb{Z}) \to H^4(Z, \mathbb{Z}))$

Sketch of Th^{m}1 (A , B , C)

- CLAIM A Φ descends to $A_X^{\mu_2} \xrightarrow{\sim} A_{X'}^{\mu_2}$
  - $\left(X_2 : \tau \simeq S_{A_X}^{-3}[k] \Rightarrow \tau \text{ commutes} \right)$ $\left(\tau[1] \simeq R^d, R = \amalg_{\mathcal{O}_X}(- \otimes \mathcal{O}_X(1)) \right)$
  - uniqueness of lifts of τ, τ' to cat. action. $(H^2(B\mathbb{Z}/2, \mathbb{C}^\times) = 0)$
- Th^m 3 : $A_X^{\mu_2} \simeq A_{X'}^{\mu_2}$ $\Rightarrow$ $H^2_{prim}(Z, \mathbb{Z}) \simeq H^2_{prim}(Z', \mathbb{Z})$
- classical Torelli for Z [Donagi '83 (X_2) / Masa-Hiko Saito '86 (Y_1)] $\Rightarrow$ $Z \simeq Z'$] B
  - $\phi : Z \simeq Z'$
  - X v. gen $\Rightarrow$ $\rho(Z) = 1 = \rho(Z')$ $\Rightarrow \phi^*(h') = h$ (ample gens)
  - global section of (multiple of) h, h' identified
  - embeddings into $\mathbb{P}$ coincide up to proj. transformation.
- $\Rightarrow$ $X \simeq X'$ C $\square$

Ingredients for Th^{m}3

• [Blanc '16, Perry '22] $K^{top}(A_X^{\mu_2})$ has a Hodge structure & Euler pairing (induced from $K^{top}([X/\mu_2])$)

what is it ? $K_0^{top}(D^b X)_{\mathbb{Q}} \overset{ch}{\simeq} H^{even}(X, \mathbb{Q})$ (Tate twist) (K_t^{top}, sm, $\oplus H^{2k-t}$)

$$\begin{array}{ccc} K_0^{top}(A_X^{\mu_2}) & \xrightarrow[\Phi^{\mu_2}\ \text{FM}]{\sim} & K_0^{top}(A_{X'}^{\mu_2}) \\ \cup & & \cup \\ K_0(A_X^{\mu_2})^{\perp} & & \\ \text{IS} & \text{Th}^m 2 + \text{H.R.R.} \ (+ K^{top} \text{ sees SODs}) & \\ K_0(D^b(Z))^{\perp} & \xrightarrow{\sim} & K_0(D^b(Z'))^{\perp} \\ \text{ch IS} & X \text{ very general } (\Rightarrow \rho(Z)=1) & \text{IS} \\ H^2_{prim}(Z, \mathbb{Z}) & \xrightarrow{\sim} & H^2_{prim}(Z', \mathbb{Z}') \end{array}$$

algebraic K-theory = Groth. group

Hodge isom.