Birational King's Conjecture

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Based on: `arXiv:2501.00130`
joint with Ballard, Berkesch, Brown, Cranton Heller, Erman, Favero, Ganatra, and Huang

Derived And NonCommutative Enthusiasts seminar
February 10, 2025

Derived category of $\mathbb{P}^n$

Let $D^b(X)$ be the bounded derived category of coherent sheaves on a variety X.

Theorem (Beilinson, 1978)

The Serre twisting sheaves $\mathcal{O}(-n), \ldots, \mathcal{O}(-1), \mathcal{O}$ form a full strong exceptional collection for $D^b(\mathbb{P}^n)$.

Full: $\mathcal{O}(-n), \ldots, \mathcal{O}(-1), \mathcal{O}$ generate $D^b(\mathbb{P}^n)$.

Exceptional:

- $\mathrm{Hom}^p(\mathcal{O}(-i), \mathcal{O}(-i)) = \begin{cases} \mathbb{k} & p = 0 \\ 0 & p \neq 0 \end{cases}$ and
- $\mathrm{Hom}(\mathcal{O}(-j), \mathcal{O}(-i)) = 0$ if $i > j$.

Strong: When $i < j$,

$$\mathrm{Hom}^p(\mathcal{O}(-j), \mathcal{O}(-i)) = 0$$

for $p > 0$.

King's conjecture

Natural to try to extend to toric varieties.

Conjecture (King, 1997)

If X is a smooth projective toric variety, then $D^b(X)$ has a full strong exceptional collection of line bundles.

Holds for: Picard rank two (Costa and Miró-Roig), Fano of dimension at most 4 (King, Borisov-Hua, Bernardi-Tirabassi, Uehara, Prabhu-Naik), and a smattering of other cases.

BUT...

King's conjecture is false

Theorem (Hille-Perling, 2006)

There is a threefold iterated toric blow-up X of the Hirzebruch surface of type 2 such that $D^b(X)$ admits no full strong exceptional collection of line bundles.

- Michalek, 2010: King's conjecture is false for $\mathbb{P}^n$ blown up at two points for n sufficiently large.
- Efimov, 2014: King's conjecture is also false for many smooth Fano toric varieties.

What do we have instead?

Theorem (Kawamata, 2006)

If X is a smooth projective toric variety, then $D^b(X)$ has a full exceptional collection.

Theorem (H-Hicks-Lazarev, 2024)

If X is a smooth toric variety, there is a length $\dim(X)$ resolution of the diagonal of X by products of line bundles.

The Bondal-Thomsen collection

The line bundles that appear in our resolution are called the Bondal-Thomsen collection Θ. $\mathcal{O}(D) \in \Theta$ if and only if one of the following equivalent conditions are satisfied:

- (Thomsen) $\mathcal{O}(D)$ is a summand of $(F_\ell)_*\mathcal{O}$ for some ℓ where ℓ is the toric Frobenius morphism.
- D is rationally equivalent to $\sum a_\rho D_\rho$ where $-1 < a_\rho \leq 0$ for all $\rho \in \Sigma(1)$.
- (Bondal) There is a $\theta \in M_\mathbb{R}$ such that

$$D = \sum_{\rho \in \Sigma(1)} \lfloor -\langle \theta, u_\rho \rangle \rfloor D_\rho.$$

The Bondal-Thomsen collection example

A notable feature

The Bondal-Thomsen collection and the resolutions we construct only depend on $\Sigma(1)$!
$\Sigma(1)$ does not correspond to X but rather a finite set of toric varieties parameterized by the secondary or GKZ fan $\Sigma_{GKZ}(X)$.
Of particular importance to us will be

$$X_1 = X, \ldots, X_k$$

corresponding to the chambers of $\Sigma_{GKZ}(X)$.

A new category

Let $\widetilde{\mathcal{X}}$ be any smooth toric Deligne-Mumford stack with birational toric morphisms $\pi_i \colon \widetilde{\mathcal{X}} \to \mathcal{X}_i$ for all i.

Definition

Let $D_{\mathrm{Cox}}(X)$ be the full subcategory of $D(\widetilde{\mathcal{X}})$ generated by $\pi_i^* D(\mathcal{X}_i)$. Further, for $\mathcal{O}_X(D) \in \theta$, let $\mathcal{O}_{\mathrm{Cox}}(D) = \pi_i^* \mathcal{O}(D)$ for any i such that $-D \in \Gamma_i$.

Long live the King's conjecture

Theorem (BBBCHEFGHH)

If X is a projective toric variety, the $\mathcal{O}_{\mathrm{Cox}}(D)$ in Θ are a full strong exceptional collection with respect to any linearization of the partial order $D \leq D'$ if and only if $D' - D$ is effective.

How?

- Fullness comes from HHL resolution depends only on $\Sigma(1)$.
- SEC from Θ-transform lemma:

Lemma (Θ-transform lemma)

For any j,

$$\pi_{j*}\mathcal{O}_{\mathrm{Cox}}(D) = \mathcal{O}_{\mathcal{X}_j}(D).$$

Proving SEC

Sample computation

Connection with mirror symmetry

Connection with mirror symmetry

The end

Thank you!