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§1 Background

$\mathcal{L}$ a line bundle on a proper variety X.

$\mathcal{L}$ is ample $\iff$ $n \gg 0$, $X \hookrightarrow \mathbb{P}H^0(X, \mathcal{L}^{\otimes n})$ is a cld. imm.

$\mathcal{L}$ is big $\iff$ $n \gg 0$, $X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{L}^{\otimes n}))$ is birat. onto its image.

$\iff \exists s \in H^0(X, \mathcal{L}^{\otimes n})$ $n \gg 0$ s.t. $\phi \neq X \setminus V(s)$ is quasi-affine.

Nakai-Moishezon criterion: $\mathcal{L}$ is ample $\iff$ $\forall Z \subseteq X$ closed subvar. $\mathcal{L}|_Z$ is big.

$\mathrm{Perf}(X)$ = derived cat. of perfect cpx'es

Fact. $\mathcal{L}$ or $\mathcal{L}^{-1}$ ample

$\Rightarrow \{\mathcal{L}^{\otimes n} \mid n \in \mathbb{Z}\}$ split-generates $\mathrm{Perf}(X)$

i.e., any perfect cpx can be obtained by taking cones, shifts, fin. direct sums, and direct summands starting from $\mathcal{L}^{\otimes n}$'s.

$\iff$ $\mathcal{L}$ is $\otimes$-generating.

Q. $\mathcal{L}$ $\otimes$-generating $\Rightarrow$ $\mathcal{L}$ or $\mathcal{L}^{-1}$ ample?

A. (I.-Olander) No. "derived cat. analogue of N.M. criterion"

$\mathcal{L}$ is $\otimes$-gen. $\iff$ $\forall Z \subseteq X$ closed subvar., $\mathcal{L}|_Z$ or $\mathcal{L}^{-1}|_Z$ is big

can vary among Z

Consequences: $\otimes$-gen. line bundles give

* well-behaved class of generators.
* the cone of corresponding $\mathbb{R}$-divisors $\leadsto$ numerical study of generation.
* more examples of varieties completely determined by Perf via (generalization of) Bondal-Orlov reconstruction.

Thm. (Bondal-Orlov, Ballard, I.) X proper Gorenstein var. / ω_X $\otimes$-gen.

① X can be canonically constructed from $\mathrm{Perf}(X)$

② $\mathrm{Perf}(X) \simeq \mathrm{Perf}(Y)$ Y variety $\Rightarrow X \cong Y$

③ $\mathrm{Auteq}(\mathrm{Perf}(X)) = \mathrm{Aut}(X) \ltimes \mathrm{Pic}(X) \times \mathbb{Z}[1]$ ← "standard autoequiv."

E.g. * X Fano (e.g. toric)

* X sm proj surface w/ ω_X big or anti-big.

  ω_X is $\otimes$-gen. $\iff$ X has no (-2)-curve.

* $\mathbb{P}_E(\mathcal{O}_E \oplus \mathcal{L})$ for E elliptic curve and $\deg \mathcal{L} \neq 0 \Rightarrow \omega_X$ $\otimes$-gen.

* Blow-ups of finitely many pts on Y w/ ω_Y ample.

Q. More example? What happens to ③ if ω_X is not $\otimes$-gen.?

§2 Toric cases.

Theorem (I.-M. Zeng-X. Zeng) X proper toric variety, $\mathcal{L}$ l.b.

$\mathcal{L}$ is $\otimes$-gen. $\iff$ $\forall Z \subseteq X$, $\mathcal{L}|_Z$ or $\mathcal{L}^{-1}|_Z$ is big.

T-invt cld subvar. ← finitely many.

$\mathcal{L}$ is big $\iff$ $\exists s \in H^0(X, \mathcal{L}^{\otimes n})$ s.t. $X \setminus V(s) = T$

$\mathcal{L} = \mathcal{O}_X(\sum_{\rho \in \Sigma(1)} a_\rho D_\rho)$ $\iff$ $P_D := \{ m \in M_{\mathbb{R}} \mid \langle m, u_\rho \rangle \geq -a_\rho, \ \forall \rho \in \Sigma(1) \}$ is full-dim'l in $M_{\mathbb{R}}$.

↑ combinatorial.

Rem. For proper Gorenstein toric var., $\omega_X^{-1} = \mathcal{O}_X(\sum_{\rho \in \Sigma(1)} D_\rho)$ is always big.

so it suffices to test with intermediate dim'l T-invt. subvariety.

E.g. $X = \mathbb{F}_n$ $(= \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-n)) \xrightarrow{C_0} \mathbb{P}^1)$

T-invt curves $\sim$ f (fiber), C_0 ($C_0^2 = -n$), $C_0 + nf$ ($(C_0+nf)^2 = n$) $\quad f^2 = 0$, $f \cdot C_0 = 1$.

$\omega_X^{-1} = 2f + C_0 + (C_0 + nf) = 2C_0 + (n+2)f$.

* $\omega_X^{-1}|_f = 2$
* $\omega_X^{-1}|_{C_0} = -2n + (n+2) = -n+2 \neq 0 \iff n \neq 2$.
* $\omega_X^{-1}|_{C_0+nf} = (-n+2) + 2n = n+2 \neq 0$

$\Bigg\}$ $\omega_{\mathbb{F}_n}$ is $\otimes$-gen. $\iff n \neq 2$.

E.g. (IZZ) Oda, Fujino-Sato $\Rightarrow$ smooth proper toric threefolds $Z''_{13}(a,b,c,d)$, $a,b,c,d \in \mathbb{Z}$.

s.t. * $a \neq c$, $b \neq 0$ $\Rightarrow$ non-proj.

* Given some numerical condition $\Rightarrow$ No nontrivial FM partner.

Claim.

* $\overset{\shortparallel}{Z_{13}}\ (a, 2, a+2, a+1)$ has $\otimes$-gen. canonical $\forall a \in \mathbb{Z}$
  non-proj.
* $\overset{\shortparallel}{Z_{\infty}}\ (a, 2, a+2, a)$ has $\otimes$-gen canonical $\iff a \neq -1, -2$.
  When $a = -1, -2$, there exist $(-1,-1)$-curves $\Rightarrow$ Atiyah flop
  $\Rightarrow \mathrm{Auteq} \supsetneq \mathrm{Auteq}^{st}$

§3 Obstruction for $\otimes$-generation of canonical.

It is also natural to ask how failure of $\otimes$-gen. influences $\mathrm{Perf}(X)$.

Def X proper variety.
$Z \subseteq X$ is obstructing ($\otimes$-gen. of canonical) if $\omega_X|_Z$ is neither big nor anti-big.

E.g. * X sm proj. surf. ω_X big or anti-big.
$C \subseteq X$ is obstructing $\iff$ C is a (-2)-curve.
($\Longrightarrow T_{\mathcal{O}_C}$ spherical twist $\Rightarrow \mathrm{Auteq} \supsetneq \mathrm{Auteq}^{st}$).

* Toric threefold.
A $(-1,-1)$-curve $C \subseteq X$ is obstructing
$\Rightarrow$ flop-flop autoeq $\Rightarrow \mathrm{Auteq} \supsetneq \mathrm{Auteq}^{st}$

Q. Does every obstructing subvariety contribute to non-standard autoeq?

Thm (IZZ) X toric threefold, $S \subseteq X$ obstructing T-inv't surface.
If $\omega_X|_S$ is nef or anti-nef, then there is an EZ diagram

$$\begin{tikzcd} S \arrow[r, hook] \arrow[d, "\text{flat (Iitaka fibration)}"] & X \\ \mathbb{P}^1 & \end{tikzcd}$$

giving a non-standard autoequivalence by EZ/Horja-twist.