Generation and Approximation in derived Cats

$\mathcal{T}$ Δ-cat with $\oplus$ Eg. $D_{qc}(X)$

Generation and Approximation

- $\{G_i\}_{i\in I} \subseteq \mathcal{T}$ is <u>generating</u> if $\forall M \in \mathcal{T}$
  $\mathrm{Hom}(G_i[m], M) = 0 \ \forall i \in I$ and $m \in \mathbb{Z} \Rightarrow M = 0$
- $C \in \mathcal{T}$ is <u>compact</u> if $\mathrm{Hom}(C, -)$ preserves $\oplus$

When $\mathcal{T} = D_{qc}(X)$

- $P \in D_{qc}(X)$ is <u>perfect</u> if P is locally $\sim$ a bd complex of vector bundle
  (locally compact). (dualizables in $D_{qc}(X)$, $\otimes^L$)
- $E \in D_{qc}(X)$ is <u>m-pseudo-coherent</u> if E is locally on X
  $m \in \mathbb{Z}$

$$\exists \underset{\text{perfect}}{P} \longrightarrow E \quad \text{s.t} \quad \left.\begin{array}{ll} H^i(P) \twoheadrightarrow H^i(E) & i = m \\ H^i(P) \xrightarrow{\sim} H^i(E) & i > m. \end{array}\right\} \text{m-isomorphism}$$

We say $E \in D_{qc}(X)$ is <u>pseudo-coherent</u> if it is m-pc $\forall m \in \mathbb{Z}$.

Pseudo-coherent = locally approximable by perfects

Consider a triple (T, E, m) on X where

- $T \subseteq |X|$ closed
- $E \in D_{qc}(X)$
- $m \in \mathbb{Z}$.

We say <u>approx by perfect</u> (resp. compact) holds for (T, E, m) if $\exists$ a perfect complex P supported on T and a map

$$P \longrightarrow E \quad \text{s.t} \ {}^{(\text{resp. compact})} \left.\begin{array}{ll} H^i(P) \twoheadrightarrow H^i(E) & i = m \\ H^i(P) \xrightarrow{\sim} H^i(E) & i > m \end{array}\right\} \text{use m-isomorph}$$

We say <u>approxima by perfect (resp. compact) holds for X</u>

if $\exists r \in \mathbb{Z}$ s.t approx by perfect (resp. compact) holds for every triple (T, E, m) where

- E is $(m-r)$-pc
- $H^i(E)$ is supported on T. $i \geq m-r$

Some questions

① $D_{qc}(X)^c \stackrel{?}{=} \mathrm{Perf}(X)$

② Is $D_{qc}(X)$ compactly generated?

③ Does approx by perfect complex holds for X?

When X is a scheme (Lipman–Neeman '07)

Assume X is qcqs

① ✓ $D_{qc}(X)^c = \mathrm{Perf}(X)$ ($\Gamma(X, -)$ commutes with $\oplus$)

② ✓ $D_{qc}(X)$ compactly generated by a single object.

③ ✓ approx by perfect holds for X.

② + ③ in this generality is due to Lipman–Neeman '07

②: Thomason, Neeman, Bondal–Van den bergh

How does one prove this?

②, ③ are local-global principles

- Induction on # affine covers
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$$\begin{tikzcd} U \cap V \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\boxtimes"] & X \arrow[d] \\ U \arrow[r, hook] & X = U \cup V \end{tikzcd}$$

gluing by Thomason localization

- Use Koszul complexes to control supports.

What about stacks

① X qcqs $\not\Rightarrow$ $D_{qc}(X)^c = \mathrm{Perf}(X)$ need "finite coh dim".

$\Leftrightarrow \mathcal{O}_X$ is compact.

E.g. $X = B\mathbb{G}_a$ in char $p > 0$

② X qcqs alg stack. Then $D_{qc}(X)$ is compactly generated

finite stab → if • X has quasifinite separated Δ (by a single)

• X admits a "good moduli space" (by a countable set)

③ infinite stab

No approx by perfects but

there is approx by compacts!

Thm (HLLP '25)

X a qcqs alg stack. Approx by compact holds for X if (1) X has quasifinite separated Δ / PM of char 0

(2) X admits a "good moduli space".

Sketch of proof

Claim: Approx by compacts is local in the quasifinite flat topology

Claim $\Rightarrow$ Thm

(1) $\exists\ \operatorname{Spec} A \xrightarrow[\text{qf sep rep}]{} X$

(2) $\exists\ X' \xrightarrow[\text{sep, rep}]{\text{étale}} X$

X' has enough compact vector bundles.

Proof of claim (sketch)

We use QFF devissage due to Hall-Rydh '18

this is a powerful tool for ~~global~~ local-global principles

Let $E = \mathrm{Stack}_{\text{rep. sep. qf. fp}/X}$ consists of $(\mathcal{V} \xrightarrow[\text{qf fp}]{\text{rep, sep}} X)$

Define $\mathrm{App} \subseteq E$ full subcat consists of those $(\mathcal{V} \to X)$

where for every étale $(\mathcal{V}' \to \mathcal{V}) \in E$ approx by compacts holds for $\mathcal{V}'$

QFF devissage $\Rightarrow$ It suffices to show.

1) if $(\mathcal{W} \to \mathcal{V}) \in E$ is étale, then $\mathcal{V} \in \mathrm{App} \Rightarrow \mathcal{W} \in \mathrm{App}$

2) if $(\mathcal{W}' \to \mathcal{W}) \in E$ is finite flat and surjective, $\mathcal{W}' \in \mathrm{App} \Rightarrow \mathcal{W} \in \mathrm{App}$

3) if $(\mathcal{W} \overset{i}{\hookrightarrow} \mathcal{V}), (\mathcal{V}' \xrightarrow{\nu} \mathcal{V}) \in E$ s.t. ν is étale and an iso on $\mathcal{V} \setminus \mathcal{W}$, i is an open immersion, and then $\mathcal{W}, \mathcal{V}' \in \mathrm{App} \Rightarrow \mathcal{V} \in \mathrm{App}$.

1) is by def'n of App ✓

3) is technically difficult but the idea is similar to

2) $\begin{array}{c} W' \quad r \\ f \Big\downarrow \text{finite flat surj} \\ W \quad r \end{array}$ f_* has a right adjoint $f^\times$ s.t. $f^\times$ is t-exact and preserves p.c.

Take a triple (T, E, m), on X. Let t be the largest integer s.t. $H^t(E) \neq 0$.

If $t < m-r$ then $\rightsquigarrow (f^{-1}(T), f^\times E, m)$ on X.

$0 \to E$ approx $\rightsquigarrow P \to f^\times E \quad f_* P \to f_* f^\times E \to E$

$\rightsquigarrow \quad f_* P \to E \to E' \to f_* P[1]$ on X

Note: E' is $(m-r)$-pseudo coherent + $H^i(E)$ supported on T. $i \geq m-r$

Claim: $H^t(E') = 0$

$$\begin{array}{ccccccc} H^t(f_* P) & \to & H^t(E) & \to & H^t(E') & \to & H^{t+1}(f_* P) \\ \| & & \| & & \| & & \cong \text{ finite flat} \\ f_* H^t(P) & \twoheadrightarrow & H^t(E) & \to & H^t(E) & \to & f_* H^{t+1}(P) = 0 \\ & & & & \| & & \\ & & & & 0 & & \end{array}$$

f finite flat

$P \to f^\times E$ is an m-isomorphism

$H^t(P) \twoheadrightarrow H^t(E)$ by assumption

$H^i(f_* f^\times E) \twoheadrightarrow H^i(E)$ by adjunction

By induction, $\exists\, Q \to E'$ approximation

$$\begin{array}{ccccccc} f_* P & \to & K & \to & Q & \to & f_* P[1] \\ \| & & \downarrow & & \downarrow & & \| \\ f_* P & \to & E & \to & E' & \to & f_* P[1] \end{array}$$

K is compact and $K \to E$ is the desired approx. $\square$

Application

- X Noetherian stack with approx by compacts $G \in D^b_{coh}(X)$ then $Perf(X) \subseteq \langle G \rangle_n \iff \langle G \rangle_n = D^b_{coh}(X)$. For alg spaces this is due to Olander–Lank '24
- $\mathcal{T}$ Δ-cat with $\oplus$ and a t-structure $\mathcal{T}_c^- = \{ E \in \mathcal{T} \mid \forall m > 0\ \exists\, \Delta$ in $\mathcal{T}$ $G \to E \to D$ s.t. $G \in \mathcal{T}^c$ and $D \in \mathcal{T}^{\leq -m} \}$ then $D_{qc}(X)^-_c = D_{pc}(X)$ pseudo-coherent