Derived McKay correspondence for some complex reflection groups of rank 2

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History

McKay (1980) demonstrated

$$\{G \underset{\text{finite}}{<} SL(2,\mathbb{C})\} \longleftrightarrow \{ADE\text{-diagrams}\}$$

"McKay Quiver": Vertices are irreps

$$G \rightsquigarrow \rho_i \xrightarrow{a_{ij}} \rho_j, \quad \mathbb{C}^2 \otimes \rho_i \simeq \bigoplus a_{ij}\rho_j$$

$$\longleftrightarrow \{ADE\text{-surface singularities}\}$$

$$G \rightsquigarrow G \curvearrowright \mathbb{C}^2 \rightsquigarrow \mathbb{C}^2/G \xleftarrow{\text{singular}} \widehat{\mathbb{C}^2/G}$$

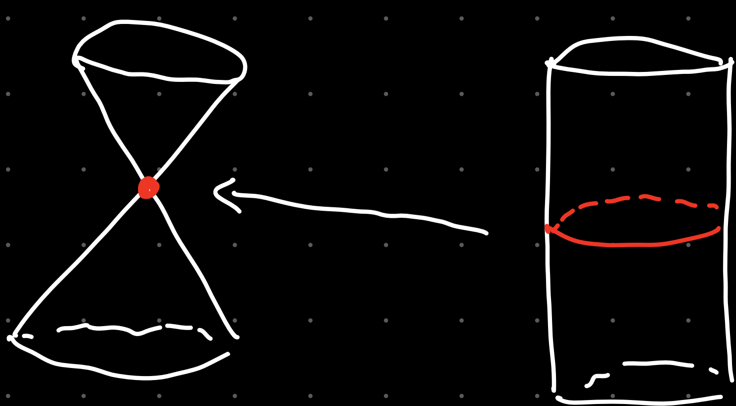
Ex: Cyclic group of order 2

$$\mathbb{Z}/2\mathbb{Z} = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle < SL(2,\mathbb{C})$$

$x \mapsto -x, \quad y \mapsto -y$

x^2, y^2, xy

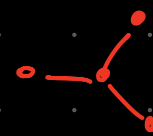
$$\rightsquigarrow \mathbb{C}^2/G = \operatorname{Spec} \mathbb{C}[x^2, y^2, xy] \simeq \operatorname{Spec} \frac{\mathbb{C}[u,v,w]}{(uv-w^2)}$$

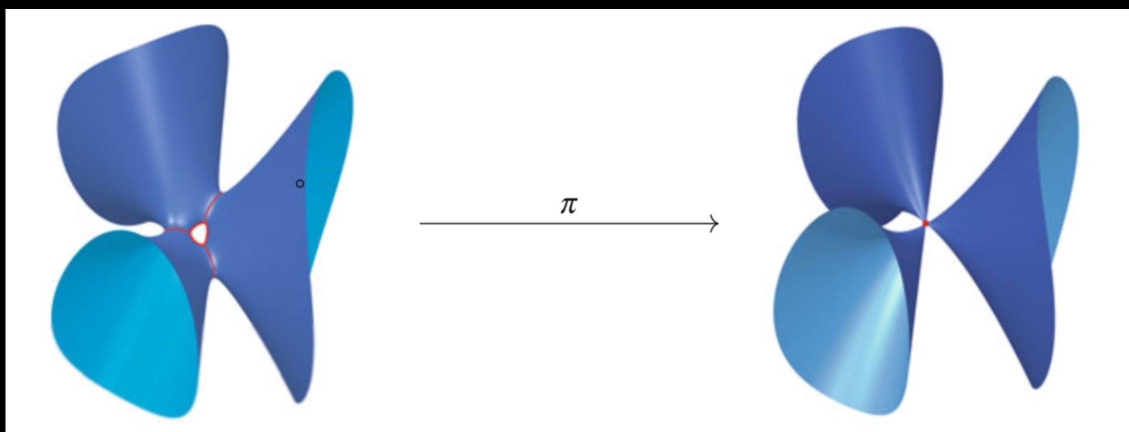


__Ex__: Binary Dihedral Group of order 8

$$G := \left\langle \begin{pmatrix} \zeta_4 & 0 \\ 0 & \zeta_4^{-1} \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\rangle < SL(2,\mathbb{C}).$$

$$\mathbb{C}^2/G = \operatorname{Spec} \mathbb{C}[u,v,w]/(u^3 + uv^2 + w^2)$$





Gonzalez - Sprinberg & Verdier ('83):

$$K_0(\widehat{\mathbb{C}^2/G}) \simeq K_{0,G}(\mathbb{C}^2)$$

Ito - Nakamura ('96, '99)

$$\widehat{\mathbb{C}^2/G} \cong G\text{-}\operatorname{Hilb}(\mathbb{C}^2)$$

$$G \curvearrowright \mathbb{C}^2 \implies G \curvearrowright \operatorname{Hilb}^{|G|}(\mathbb{C}^2)$$

Def: $G\text{-Hilb}(\mathbb{C}^2)$ is the subscheme of $\mathrm{Hilb}^{|G|}(\mathbb{C}^2)$ s.t. $\mathcal{O}_{\mathbb{C}^2}/I \simeq \mathbb{C}[G]$

$\nwarrow$ regular representation

$\Rightarrow$ Gives a moduli-theoretic interpretation!

$Z \subset G\text{-Hilb} \times \mathbb{C}^2$ be the universal family

$$\begin{tikzcd} & Z \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ G\text{-Hilb} \arrow[dr, "\text{Hilbert-Chow}"'] & & \mathbb{C}^2 \arrow[dl, "\text{quotient}"] \\ & \mathbb{C}^2/G & \end{tikzcd}$$

$\rightsquigarrow$ An equivariant integral functor

$$\Phi : D^b_G(\mathbb{C}^2) \to D^b(G\text{-Hilb})$$

$$\mathcal{E} \longmapsto R{p_1}_*^G(p_2^* \mathcal{E} \overset{L}{\otimes} \mathcal{O}_Z)$$

Theorem (Kapranov-Vasserot '00): Φ is an equivalence of categories.

"Derived McKay Correspondence"

Where to go from here?

1) Higher Dimensions. $G < SL(n,\mathbb{C})$, $G \curvearrowright \mathbb{C}^n$

Bridgeland-King-Reid ('01): $n=3$ $(+\,\varepsilon)$

no "pseudoreflections"

2) $G \underset{\text{finite}}{<} GL(2,\mathbb{C})$, G is small

Ishii-Ueda ('15): $D^b(G\text{-Hilb}) \hookrightarrow D^b_G(\mathbb{C}^2)$ as a semiorthogonal component.

Can generalize to reflection groups!

<u>Conjecture (Polishchuk-Van den Bergh):</u> Let G be a finite group acting on a smooth variety X s.t. $X^g/C(g)$ is smooth. Then $D^b_G(X) = \langle \mathcal{A}_{[g]} \mid [g] \text{ conj class} \rangle$ and $\mathcal{A}_{[g]} \cong D^b(X^g/C(g))$.

<u>Theorem (Bhaduri-Davidov-Faber-Honigs-McDonald-Walker-S):</u> For $X = \mathbb{C}^2$, $G \in \{G(m,m,2), G(2m,m,2), G_{12}, G_{13}, G_{22}\}$, then the conjecture holds

<u>Reflection Groups</u>

= pseudoreflections.

A complex reflection $r \in GL(n,\mathbb{C})$ is an element of finite order that pointwise fixes a hyperplane.

A complex reflection group is a subgroup of $GL(n,\mathbb{C})$ which is generated by reflections.

If G is a reflection group acting on $\mathbb{C}^n$, its rank is $\dim((\mathbb{C}^n)^G)^\perp$.

For our groups $\{G(m,m,2), G(2m,m,2), G_{12}, G_{13}, G_{22}\}$

$$G \rightsquigarrow \underbrace{H}_{\text{index 2 in } G} = G \cap SL(2,\mathbb{C})$$

all finite subgrps of $SL(2,\mathbb{C})$ up to conjugacy

G	H	
$G(m,m,2)$	$\mathbb{Z}/m\mathbb{Z}$	A_m
$G(2m,m,2)$	BD_{4m}	D_m
G_{12}	$B\,\text{Tetra}$	E_6
G_{13}	$B\,\text{Octa}$	E_7
G_{22}	$B\,\text{Ico}$	E_8

$A := G/H \simeq \mathbb{Z}/2\mathbb{Z}$

$$\begin{tikzcd}
G \curvearrowright \mathbb{C}^2 \arrow[d, "\text{quotient}"] & \\
A \curvearrowright \mathbb{C}^2/H \arrow[d, "\text{quotient}"] & H\text{-Hilb} \curvearrowleft A \arrow[l, "\text{res}"'] \arrow[d, "\text{quotient}"] \\
\mathbb{C}^2/G & H\text{-Hilb}/A \simeq \mathrm{Bl}_{p_1,\dots,p_m} \mathbb{C}^2/G \arrow[l, "\text{blow-up}"']
\end{tikzcd}$$

<u>Lemma</u>: H-Hilb$/A$ is smooth

Semiorthogonal Decompositions of Equivariant Derived Categories

Recall, given a triangulated category T, a semiorthogonal decomposition consists of full triangulated subcategories $\mathcal{A}_1, \dots, \mathcal{A}_n$ satisfying:

1) $\mathrm{Hom}(\mathcal{A}_i, \mathcal{A}_j) = 0$ if $i > j$
2) for all $F \in T$, there is:

$$\begin{tikzcd}
0 = F_n \arrow[rr] & & F_{n-1} \arrow[dl, dashed, "{[1]}"] \arrow[r] & \cdots \arrow[r] & F_1 \arrow[rr] & & F_0 = F \arrow[dl, dashed, "{[1]}"] \\
& \mathrm{Cone}(F_n \to F_{n-1}) \arrow[ul, dashed] & & & & \mathrm{Cone}(F_1 \to F_0) \arrow[ul, dashed] &
\end{tikzcd}$$

$\mathrm{Cone}(F_i \to F_{i-1}) \in \mathcal{A}_i$

Notation: $T = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$.

Ex $D^b(\mathbb{P}^n) = \langle \mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(n) \rangle$

Exceptional: $\mathrm{Hom}(E, E[i]) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & \text{else} \end{cases}$

$\iff \langle E \rangle \simeq D^b(\mathbb{C}\text{-v.s.})$

$\underline{\text{Ex}}$: $X \to Y$ a blow-up of smooth varieties with smooth center $Z \subset Y$. $c = \operatorname{codim} Z$

$$D^b(X) = \langle D^b(Z)(1-c), \dots, D^b(Z)(-1), D^b(Y) \rangle$$

So what is an equivariant sheaf?

Let G be a finite group acting on a variety X.

$\rightsquigarrow$ Collection of autoequivalences

$$g^* : D^b(X) \xrightarrow{\sim} D^b(X) \qquad (\operatorname{coh} X)$$

An equivariant object is a pair $(F, \{\psi_g\})$

$$F \in D^b(X), \qquad \psi_g : F \xrightarrow{\sim} g^* F$$

$\psi_1 = \mathrm{id}$, $\psi_{gh} = h^*(\psi_g) \circ \psi_h$. Morphisms are ordinary morphisms which commute w/ the data

$\underline{\text{Ex}}$: $X = \operatorname{Spec} \mathbb{C}$. A G-equivariant sheaf is a representation of G!

$$D^b_G(\operatorname{Spec} \mathbb{C}) \simeq D^b(\operatorname{rep} G) \simeq \bigoplus \underbrace{D^b(\rho_i)}_{\simeq D^b(\mathbb{C})}$$

Equivalently, it is a sheaf on the quotient stack $[X/G]$.

$$\rightsquigarrow D^b_G(X) \cong D^b(\mathrm{coh}_G X) \cong D^b([X/G])$$

So how do we obtain a S.O.D. of $D^b_G(X)$?

Canonical Stack:

"Normalization"

$\mathcal{X}^{can}$ ($= \mathcal{X}$ if smooth)

Root Stack: $\mathcal{X}[\sqrt[r]{D}]$

$D = (D_1, \dots, D_n)$, $r = (r_1, \dots, r_n)$

$$\begin{tikzcd} \mathcal{X}[\sqrt[r]{D}] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & {[\mathbb{A}^n/\mathbb{G}_m^n]} \arrow[d, "\wedge r"] \\ \mathcal{X} \arrow[r, "D"'] & {[\mathbb{A}^n/\mathbb{G}_m^n]} \end{tikzcd}$$

"r^{th} roots of D"

Theorem (Geraschenko - Satriano): Let $\mathcal{X}$ be a smooth, separated, tame DM stack with coarse moduli space Y. Let $\mathcal{D}$ be the branch divisor of $\mathcal{X} \to Y$ and D the pullback to Y^{can}. Then

$$\begin{tikzcd} \mathcal{X} \cong (Y^{can}[\sqrt[r]{D}])^{can} \arrow[r] \arrow[rrr, bend right, "\text{coarse moduli}"'] & Y^{can}[\sqrt[r]{D}] \arrow[r] & Y^{can} \arrow[r] & Y \end{tikzcd}$$

Corollary: $[X/G] \to (X/G)^{can}$ is a root stack along smooth snc divisors.

Theorem (Ishii-Ueda, Bergh-Lunts-Schnürer): There is an S.O.D.: $(D = (D_i))$

$$D^b(\mathcal{X}[\sqrt[r]{D}]) = \langle D^b(D)(r-1), \ldots, D^b(D)(1), D^b(\mathcal{X})\rangle$$

Gives a prescription for S.O.D.'s of equivariant categories.

$G \curvearrowright X$, G finite, X = quasi-proj.

$\Rightarrow [X/G] \longrightarrow (X/G)^{can}$ root stack

$D = \sum D_i$, $r_i = |\mathrm{Stab}(D_i)|$

dim = 2, another SOD is known (Ishii-Ueda)

$$D^b_G(X) = \langle \{D^b(D_1)\}_{i=1}^{r_1-1}, \ldots, \{D^b(D_n)\}_{i=1}^{r_n-1}, D^b((X/G)^{can})\rangle$$

For us: $X/G \cong (X/G)^{can}$

$G \cap SL(2,\mathbb{C})$

Theorem (Ishii-Ueda): $D^b_G(\mathbb{C}^2) \cong D^b_A(H\text{-Hilb})$

$A \cong \mathbb{Z}/2\mathbb{Z}$

$$[H\text{-Hilb}/A] \longrightarrow \overbrace{H\text{-Hilb}/A}^{\text{smooth}}$$

root stack

Branch div $D_1, \ldots, D_n$, $r_i = 2$

blow-ups

$$D^b_G(\mathbb{C}^2) = \langle D^b(D_1), \dots, D^b(D_n), D^b(H\text{-Hilb}/A) \rangle$$

$$= \langle D^b(D_1), \dots, D^b(D_n), \underbrace{E_1, \dots, E_m}_{\text{exceptional}}, D^b(\mathbb{C}^2) \rangle$$

$D^b(\mathbb{C}^2/G)$

$G(2m, m, 2)$ case m even:

$A \curvearrowright H\text{-Hilb}$