Stringy Kähler moduli of flops via GIT

DANCE Seminar

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1 Flops

Setting

- Fix a map $f : X \to \operatorname{Spec} R$ where X is a complex quasi-projective 3-fold.
- The map contracts a curve $C \cong \mathbb{P}^1$ to a point.
- Crucially, $f^{-1}(0)$ is non-reduced.
- The **length** of the flopping contraction is the multiplicity of C in $f^{-1}(0)$.
- **Theorem** [Katz-Morrison '92]: The length l is such that $1 \leq l \leq 6$ and all values can be obtained.

Setting

- Fix a map $f : X \to \operatorname{Spec} R$ where X is a complex quasi-projective 3-fold.
- The map contracts a curve $C \cong \mathbb{P}^1$ to a point.
- [Atiyah flop, $l = 1$] $X = \mathcal{O}(-1)^{\oplus 2}_{\mathbb{P}^1} \to \{x^2 + y^2 = z^2 + w^2\}$
- [Laufer flop, $l = 2$] $X \to \{x^2 + y^3 = tz^2 + yt^3\}$.

Flop-flop autoequivalence

$$\begin{array}{ccc} X & \dashrightarrow & X' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & \operatorname{Spec} R & \end{array}$$

- [Bridgeland '00] showed that the birational map $X \dashrightarrow X'$ induces a derived equivalence $D^b(X) \simeq D^b(X')$.
- There is a *flop-flop autoequivalence* $D^b(X) \to D^b(X') \to D^b(X)$.

Spherical Twists

- The flop-flop can be understood as a "twist" around the curve C.
- Illustrative example: for a divisor D, one can take the inverse twist around D by tensoring with $\mathcal{O}(-D)$. Note there is a short exact sequence

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0.$$

- For the Atiyah flop, there is a resolution

$$0 \to \mathcal{O}(1) \to \mathcal{O}^{\oplus 2} \to \mathcal{O}(-1) \to \mathcal{O}_C(-1) \to 0.$$

- The flop-flop will send the line bundle $O(-1)$ to the complex $\mathcal{O}(1) \to \mathcal{O}^{\oplus 2}$.

Spherical Twists

- For the Atiyah flop, the sheaf $\mathcal{O}_C(-1)$ is a *spherical object* [Seidel–Thomas ''00].
- [Anno-Logvinenko '13] The modern way to phrase the autoequivalence is to take a spherical functor

$$F : D^b(A) \to D^b(X).$$

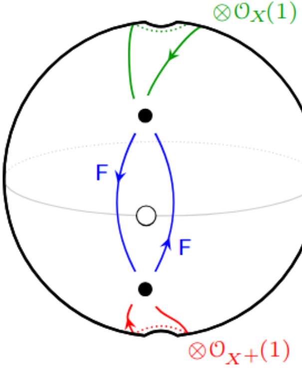
- The twist around F is $\mathrm{Cone}(FR \to \mathrm{id})$.
- In the Atiyah flop case, $F : D^b(\mathbb{C}) \to D^b(X)$.
- **I will write down autoequivalences by writing down a functor.**

2 Stringy Kähler moduli space

SKMS for the Atiyah flop

- In mirror symmetry, the SKMS is "the moduli space of complex structures of the mirror."
- Loops give us autoequivalences of $\mathrm{Fuk}\left(\hat{X}\right) \simeq D^b(X)$.
- [Aspinwall '03] used physical ideas to describe the SKMS for the Atiyah flop.

SKMS for the Atiyah flop (ii)



(Image from [Donovan–Wemyss '19])

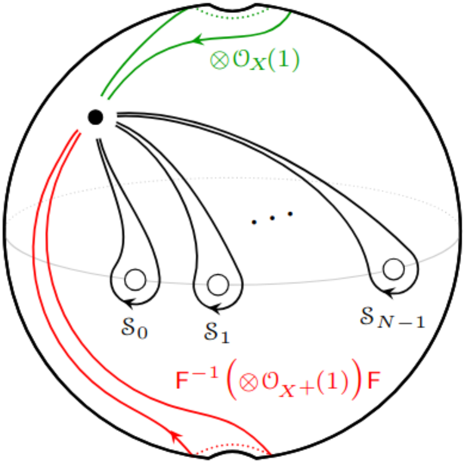
The SKMS in higher lengths

- [Toda '08] made this picture rigorous using Bridgeland stability, for flops of length 1.
- This was extended by [Hirano–Wemyss, '19] to flops of any length.
- The idea is to take

$$\mathcal{D} = \{\mathcal{F} \in D^b(X) \mid \operatorname{supp} \mathcal{F} \subset C\}$$

and define the SKMS as $\operatorname{Stab} \mathcal{D} / \operatorname{Aut} \mathcal{D}$.

The SKMS in higher lengths



- In general, there are N punctures on the equator, where N depends only on the length l of the flop as follows.

ℓ	1	2	3	4	5	6
N	1	2	4	6	10	12

(Images taken from [Donovan–Wemyss '19])

Autoequivalences

(Following the work of Donovan–Wemyss)

- The twist around the first puncture is:
  - The flop-flop;
  - The twist around C;
  - Twist around $D^b(A_{\text{con}}) \to D^b(X)$ where A_{con} is the "contraction algebra".
- The twist around the last puncture is:
  - The twist around lC;
  - Twist around $D^b(A_{\text{fib}}) \to D^b(X)$ where A_{fib} is the "fibre algebra".
- The contraction and fibre algebras are finite dimensionals possibly non-commutative, representing the deformations of C and lC respectively.

3 Geometric Invariant Theory

GIT for the Atiyah flop

- The Atiyah flop arises as a GIT quotient of a stack $\mathfrak{X}$:

$$X_+ \hookrightarrow [\mathbb{A}^4/\mathbb{C}^*] \hookleftarrow X_-$$

  where the action is $\lambda \cdot (x, y, z, w) = (\lambda x, \lambda y, \lambda^{-1} z, \lambda^{-1} w)$.
- The two quotients arise by deleting $(x = y = 0)$ and $(z = w = 0)$ respectively.

Windows

[Herbst–Hori–Page '08, Segal '11, Halpern–Leistner '15, Ballard–Favero–Katzarkov '12]

- Writing $X_+ = \mathcal{O}(-1)_{\mathbb{P}^1}^{\oplus 2}$, it is clear that $D^b(X_+) = \langle \mathcal{O}(a), \mathcal{O}(a+1) \rangle$.
- These bundles also exist on the stack $\mathfrak{X}$.
- We call $\mathcal{W}_a := \langle \mathcal{O}(a), \mathcal{O}(a+1) \rangle \subset D^b(\mathfrak{X})$ a **window**.
- Restriction to the open set X_+ induces a derived equivalence.

Window shifts

- If W is a **self-dual** representation of G, then window subcategories are derived equivalent to all GIT quotients.
- This gives us **window-shift autoequivalences.**

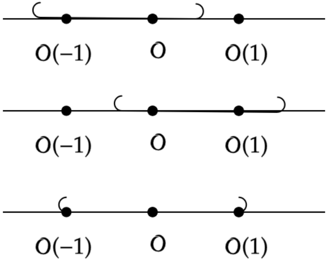
$$
\begin{array}{ccccccc}
 & \mathcal{W}_a & & & \mathcal{W}_{a+1} & & \\
\nearrow & & \searrow & \nearrow & & \searrow & \\
X_+ & & & X_- & & & X_+
\end{array}
$$

Window shifts as spherical twists

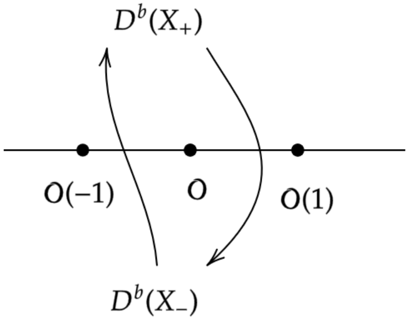
- These can be written as spherical functors with source category given by the subcategory of $D^b(\mathfrak{X})$ generated by some sheaves on the unstable locus for X_-.
- In the Atiyah flop case, we can take $\langle \mathcal{O}_{z=w=0} \rangle \subset D^b(\mathfrak{X})$.
- One can compute $\mathrm{End}_{\mathfrak{X}}(\mathcal{O}_{z=w=0}) \cong \mathbb{C}$.
- So we get a spherical functor $D^b(\mathbb{C}) \to D^b(X_+)$, recovering the classical story.

SKMS using windows

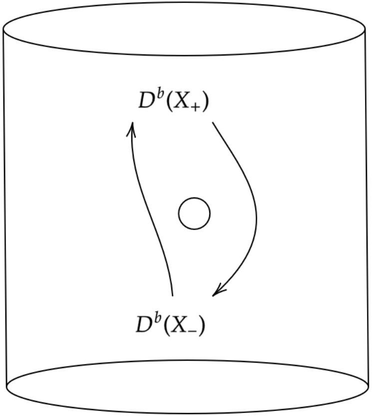
- [Halpern-Leistner–Sam '16, Spenko–Van den Bergh '19] defined a SKMS for any GIT quotient of a self-dual representation of a reductive group G using windows.



SKMS using windows (ii)



SKMS using windows (iii)



4 Two things give you the same answer

Universal length 2 flop

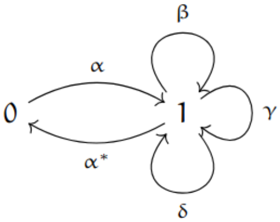
- For any length flop, there is a “universal” flop.

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{U}_l \\ \downarrow & & \downarrow \\ \Delta & \longrightarrow & \mathfrak{h} \end{array}$$

- e.g. the universal flop of length 1 is the Atiyah flop.
- Derived autoequivalences of a threefold flop should be induced by those of the universal flop.

Theorems

- [V '25] Theorem I: The universal length 2 flop can be obtained as a GIT quotient of the form $[W/G]$, where $G = \mathrm{GL}_2$ and W is a self-dual representation.
  - Based on work by [Karmazyn '17], using quiver GIT.



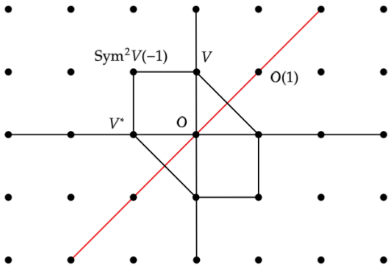
$$\alpha\alpha^* = te_0, \qquad \beta^2 = T_0^{\beta} e_1,$$

$$\gamma^2 = T_0^{\gamma} e_1, \qquad \delta^2 = T_0^{\delta} e_1,$$

$$\alpha^*\alpha + \beta + \gamma + \delta = \frac{t}{2} e_1$$

Theorems (ii)

- Theorem II: The associated SKMS is a sphere with 2 north and south pole punctures, and 2 equatorial punctures.



Theorems (iii)

- Theorem III: Monodromy around the equatorial punctures acts as twists around *universal* contraction and fibre algebras.