Derived characterization of singularities
(joint w/ Pat Lank, Peter McDonald)

Goal: Characterize classical singularities such as Du Bois, rational singularities in terms of generation stmts in $D^b_{coh}(X)$.

§1. Du Bois & rational singularities

X smooth variety$/\mathbb{C}$ $\Rightarrow$ de Rham complex

$$\Omega^\bullet_X := [\mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \cdots \to \Omega^n_X]$$

Ω^p_X = Graded pieces

$$\underline{\mathbb{C}}_X \to \Omega^\bullet_X$$

X proper: Hodge-dR s.s. $E_1^{p,q} = H^q(X, \Omega^p_X)$

$$\Downarrow$$

$$H^{p+q}_{sing}(X, \mathbb{C})$$

degenerates at E_1.

$\rightsquigarrow$ Hodge decomposition

$$H^k_{sing}(X, \mathbb{C}) = \bigoplus_{p+q=k} H^q(X, \Omega^p_X)$$

X singular $\underset{\wedge}{\Rightarrow}$ Have de Rham complex
Still

Not nice!

(Deligne, Du Bois) Substitute : Du Bois complex $\underline{\Omega}^\bullet_X$

$\underline{\mathbb{C}}_X \to \underline{\Omega}^\bullet_X$ resolution

$\underline{\Omega}^\bullet_X$ comes w/ a filtration

$\rightsquigarrow$ Graded pieces $\underline{\Omega}^p_X$. Typically not a sheaf

$\underline{\Omega}^p_X \in D^b_{coh}(X)$

Unfortunate notation!

X proper : $E_1^{p,q} = \mathbb{H}^q(X, \underline{\Omega}^p_X) \Rightarrow H^{p+q}_{sing}(X, \mathbb{C})$ degenerates at E_1.

Deligne used this to construct the mixed Hodge structure on $H^k_{sing}(X, \mathbb{C})$.

Rmk: • $n = \dim X$,

$\underline{\Omega}^n_X = Rf_* \omega_{\tilde{X}}$, $f : \tilde{X} \to X$ res. of sing.

$\omega_{\tilde{X}}$ = Canonical line bundle.

• Natural map

$$\Omega^\bullet_X \longrightarrow \underline{\Omega}^\bullet_X \quad \text{(respect. filt.)}$$

$$\rightsquigarrow \Omega^p_X \longrightarrow \underline{\Omega}^p_X \quad \forall p$$

e.g. When X smooth, these are iso $\forall p$.

Idea: Measure singularities of X by measuring how much these two complexes agree.

Defn: (Steenbrink '83) X has Du Bois sing. (DB) if:

$$\mathcal{O}_X \longrightarrow \underline{\Omega}^0_X \text{ iso.}$$

Alternate defns:

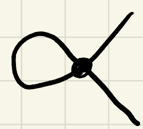
(1) (Kovács '99)

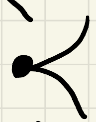
X DB sing $\iff$ $\mathcal{O}_X \to \underline{\Omega}^0_X$ splits in $D^b_{coh}(X)$

(2) (Schwede '07) $X \hookrightarrow \underset{\text{smooth}}{Y}$.

Take $\tilde{Y} \to Y$ log resolution of (Y, X) isom. outside X, $E = \pi^{-1}(X)$ simple normal crossings divisor. Then

$$\underline{\Omega}^0_X \simeq R\pi_* \mathcal{O}_E$$

Ex: (1) Nodal curve is DB 

Cuspidal curve is not DB 

(2) DB sing. come up in moduli theory, MMP

Defn: X has rational sing. if

$$\mathcal{O}_X \to Rf_*\mathcal{O}_{\tilde{X}} \text{ iso.}$$

$f : \tilde{X} \to X$ res. of sing.

Alternate defn.:

(1) (Kovács '00) X has rat'l sing

$\iff \mathcal{O}_X \to Rf_*\mathcal{O}_{\tilde{X}}$ splits in $D^b_{coh}(X)$

(2) Aside: $Rf_*\mathcal{O}_{\tilde{X}} \simeq \mathbb{D}_X(\underline{\Omega}^n_X)$ Groth. dual

$$\left[\mathcal{O}_X \to \mathbb{D}_X(\underline{\Omega}^n_X)\right] \quad (\simeq Rf_*\omega_{\tilde{X}})$$

"Dual" defn. of DB sing.

Ex: (Of rat'l sing) Toric varieties, quotient sing., klt sing. (Elkik)

Defn (Rational pairs)

- A pair (Y, I^c) Y = variety, I = ideal sheaf
  $c \in \mathbb{R}_{\geq 0}$

  $f: \tilde{Y} \to Y$ be a log res.

  $I \cdot \mathcal{O}_{\tilde{Y}} = \mathcal{O}_{\tilde{Y}}(-G)$ G = snc divisor

(Schwede-Takagi '08)

(Y, I^c) ST-rat'l pair if $\mathcal{O}_Y \xrightarrow{} Rf_* \mathcal{O}_{\tilde{Y}}(\lfloor cG \rfloor)$
iso

- A pair (Y, D) D = Weil divisor w/ coeff = 1
  $f: \tilde{Y} \to Y$ thrifty resolution.

(Kollár-Kovács '13)

(Y, D) KK-rat'l pair if $\mathcal{O}_Y(-D) \to Rf_* \mathcal{O}_{\tilde{Y}}(-D_Y)$
iso

(where D_Y = strict transform.

§ 2. Generation in derived categories

Defn: Let S be a subcatory of T,
triangulated category

(1) A triangulated subcategory of T is said to be <u>thick</u> if it is closed under direct summands.

(2) The smallest thick subcategory containing S in T is denoted by $\langle S \rangle$.
$\langle S \rangle$ has an increasing filtration by subcategories $\langle S \rangle_n$,
$\langle S \rangle_n$ obtained by shifts, finite direct sums, direct summands, taking at most n cones.

Defn: E, G objects in $\mathcal{T}$, $\mathcal{C}$ subcategory of $\mathcal{T}$

(1) E is finitely built by $\mathcal{C}$ if $E \in \langle \mathcal{C} \rangle$.

$\text{level}^{\mathcal{C}}(E)$ = Smallest integer n s.t.

$$E \in \langle \mathcal{C} \rangle_n$$

(2) A classical generator for $\mathcal{T}$ is an object $G \in \mathcal{T}$ s.t. $\langle G \rangle = \mathcal{T}$

(3) A strong generator for $\mathcal{T}$ is an object $G \in \mathcal{T}$ s.t. $\langle G \rangle_n = \mathcal{T}$ for some $n \geq 0$

(4) The Rouquier dimension of $\mathcal{T}$ denoted $\dim \mathcal{T}$ is the smallest integer d s.t. $\langle G \rangle_{d+1} = \mathcal{T}$ for some object G in $\mathcal{T}$.

<u>Conj</u>: (Orlov '09) X smooth quasi proj over a field k, then

$$\dim D^b_{coh}(X) = \dim X.$$

<u>Rmk</u>: • (Olander '23) X quasi-affine Noeth. regular scheme of dim n, then

$$\langle \mathcal{O}_X \rangle_{n+1} = D^b_{coh}(X)$$

$$\dim D^b_{coh}(X) = n$$

• (Rougier '08) X sm. quasi proj over a field k, L very ample line. bun. on X

$$\langle \bigoplus_{i=0}^{\dim X} L^i \rangle_{2\dim X+1} = D^b_{coh}(X)$$

$$\dim D^b_{coh}(X) \leq 2 \dim X$$

• (BILMP '23) X quasi-proj. over a perfect field of char >0, L = v.a. line bun.

$F^e_*(\bigoplus_{i=0}^{\dim X} L^i)$ is a strong generator for $D^b_{coh}(X)$, F = Frobenius. $e \gg 0$

§3. Main results

Thm: (Lank-V '25) $\pi : Y \to X$ proper surj w/ Y rat'l sing. TFAE

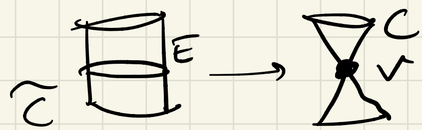
(1) X has rat'l sing

(2) $\mathrm{level}^{R\pi_* D^b_{coh}(Y)}(\mathcal{O}_X) \leq 1$

$$\mathcal{O}_X \simeq R\pi_* \mathcal{O}_Y$$

Prop: (Lank-V '25) $X \subseteq \mathbb{P}^n$ sm. hypersurface of deg d.

$C \subseteq \mathbb{A}^{n+1}$ affine cone



$$\begin{tikzcd} X \simeq E \arrow[r] \arrow[d] & \widetilde{C} \arrow[d, "\pi"] \\ v \arrow[r, hook] & C \end{tikzcd}$$

$d \geq n$, then

$$\mathrm{level}^{R\pi_* D^b_{coh}(Y)}(\mathcal{O}_X) \leq 1 + 2(d-n)$$

Thm: (Lank - V '25) $X \hookrightarrow Y$ smooth

$\pi : \tilde{Y} \to Y$ log res. $E = \pi^{-1}(X)_{red}$

TFAE.

(1) X has DB sing

(2) $\text{level}^{R\pi_* D^b_{coh}(E)}(\mathcal{O}_X) \leq 1$

(Dey '24) Has similar results in this directions.

$$\mathcal{O}_X \to R\pi_* \mathcal{O}_Y$$

Thm: (Lank - McDonald - V. '25)

- $f : \tilde{Y} \to Y$ log res of (Y, I^c)

(Y, I^c) is ST-rat'l pair $\iff$

$$\mathcal{O}_Y \in \langle Rf_* \mathcal{O}_{\tilde{Y}}(\lfloor cG \rfloor) \rangle_1$$

<u>Thm</u>: $f: \tilde{Y} \to Y$ thrifty res (Y, D)

(Y, D) is KK-dat'l $\iff$ $\mathcal{O}_Y(-D) \in \langle Rf_* \mathcal{O}_{\tilde{Y}}(-D_Y) \rangle_1$